PHY 3320 – Intermediate Classical Mechanics

Assignment 5 – Solutions

① A particle of mass m is constrained to move on the surface of a sphere of radius R by an applied force $\vec{F}(\theta,\phi)$. Write the equation of motion.

In spherical coordinates, the general expression for the acceleration is

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\phi}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\sin\theta\cos\theta\,\dot{\phi}^2)\hat{e}_\theta$$
$$+ (2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta + r\sin\theta\,\ddot{\phi})\hat{e}_\phi$$

Since we are given that the particle is constrained to move on the surface of a sphere, we have:

$$r(t) = R \quad , \quad \dot{r}(t) = 0 \quad , \quad \ddot{r}(t) = 0$$

so
$$\vec{a} = (-R\dot{\theta}^2 - R\sin^2\theta\,\dot{\phi}^2)\hat{e}_r + (R\ddot{\theta} - R\sin\theta\cos\theta\,\dot{\phi}^2)\hat{e}_\theta$$
$$+ (2R\dot{\theta}\dot{\phi}\cos\theta + R\sin\theta\,\ddot{\phi})\hat{e}_\phi$$

Note that there is a radially inward component of the acceleration, even though there's no motion in the radial direction!

The force $\vec{F}(\theta,\phi)$ can be written in components with the spherical unit vectors:

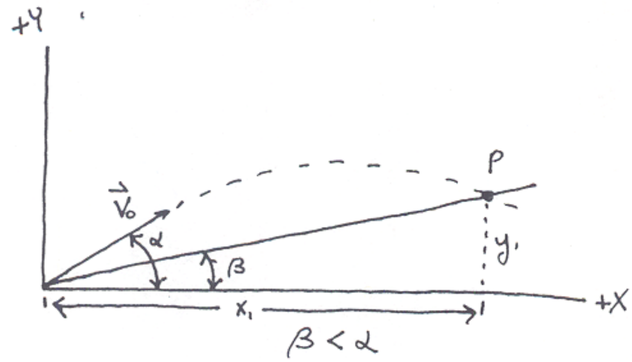
$$\vec{F}(\theta,\phi) = F_r(\theta,\phi)\hat{e}_r + F_\theta(\theta,\phi)\hat{e}_\theta + F_\phi(\theta,\phi)\hat{e}_\phi$$

By applying $\vec{F} = m\vec{a}$ for each component, we get

$$\left.\begin{aligned} &\boxed{1}\quad F_r = -mR(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) \\ &\boxed{2}\quad F_\theta = mR(\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2) \\ &\boxed{3}\quad F_\phi = mR(2\dot{\theta}\dot{\phi}\cos\theta + \sin\theta\,\ddot{\phi}) \end{aligned}\right\} \text{Equations of motion}$$

Given explicit functions for $F_\theta(\theta,\phi)$ and $F_\phi(\theta,\phi)$, one can (in principle) use equations $\boxed{2}$ and $\boxed{3}$ to obtain $\theta(t)$ and $\phi(t)$.

(Eq. $\boxed{1}$ doesn't provide any new information – it just expresses the constraint on $\vec{F}$ that is needed to keep the particle on the surface.)

② 

Calculate the time required for the projectile to cross a line passing through the origin and making an angle β with respect to the horizontal.

$$\Sigma \vec{F} = m\vec{a} \qquad \left.\begin{array}{l} \dot{x}(0) = V_0 \cos\alpha, \; x(0) = 0 \\ \dot{y}(0) = V_0 \sin\alpha, \; y(0) = 0 \end{array}\right\} \text{initial conditions}$$

X-direction: $\ddot{x} = 0, \quad \dot{x} = V_0 \cos\alpha, \quad x = V_0 t \cos\alpha$

Y-direction: $\ddot{y} = -g, \quad \dot{y} = V_0 \sin\alpha - gt, \quad y = V_0 t \sin\alpha - \frac{1}{2} g t^2$

We know that the projectile is at P when

$$\frac{y(t_1)}{x(t_1)} = \frac{y_1}{x_1} = \tan\beta .$$

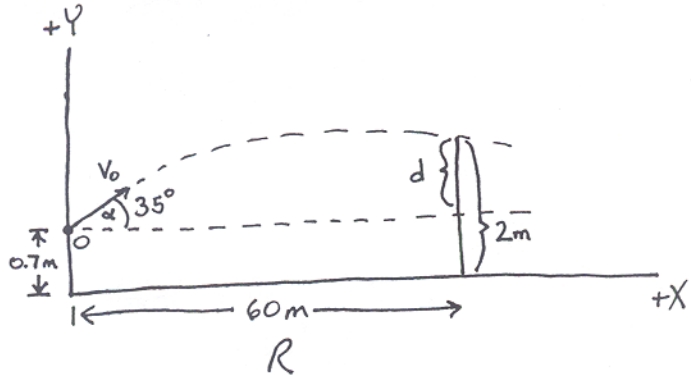
So

$$\tan\beta = \frac{V_0 t_1 \sin\alpha - \frac{1}{2} g t_1^2}{V_0 t_1 \cos\alpha} = \tan\alpha - \frac{g t_1}{2 V_0 \cos\alpha}$$

Solving for t_1:

$$t_1 = \frac{2 V_0 \cos\alpha}{g} \left[\tan\alpha - \tan\beta\right]$$

③



In order to clear the fence, what is the initial speed of the softball?

To just clear the fence, we need to find V_0 such that $y(T)=d$ and $x(T)=R$ at some time T.

Initial conditions: $\dot{x}(0)=V_0\cos\alpha$, $x(0)=0$

$\dot{y}(0)=V_0\sin\alpha$, $y(0)=0$

As in problem ②,

$$x(t)=V_0 t\cos\alpha \quad \text{and} \quad y(t)=V_0 t\sin\alpha-\tfrac{1}{2}gt^2$$

At time T:

$$x(T)=V_0T\cos\alpha=R,\quad y(T)=V_0T\sin\alpha-\tfrac{1}{2}gT^2=d$$

So, $T=\dfrac{R}{V_0\cos\alpha}$.

$$V_0\left(\frac{R}{V_0\cos\alpha}\right)\sin\alpha-\frac{1}{2}g\left(\frac{R}{V_0\cos\alpha}\right)^2=d$$

$$R\tan\alpha-\frac{gR^2}{2V_0^2\cos^2\alpha}=d$$

$$\Longrightarrow V_0=\left[\frac{gR^2}{2\cos^2\alpha\,(R\tan\alpha-d)}\right]^{1/2}$$

For $\alpha=35°$, $R=60m$, $g=9.8\ m/s^2$, $d=2m-0.7m=1.3m$:

$$V_0\approx 25.4\ m/s$$

④ A particle of mass m is subject to a force

$$F(x,t) = -kx + F_0 \sin\omega t \qquad \text{(in one dimension)}$$

$k>0$, $x(0) = X_0$, $\dot{x}(0) = V$. $\qquad \omega^2 = \frac{k}{m}$

Find $x(t)$ for $t>0$.

$F(x,t) = m\ddot{x}$ by Newton's 2nd Law.

$$\Longrightarrow m\ddot{x} = -kx + F_0 \sin\omega t, \quad \ddot{x} + \underbrace{\frac{k}{m}}_{\omega^2} x = \frac{F_0}{m}\sin\omega t$$

so

$$\ddot{x} + \omega^2 x = \frac{F_0}{m}\sin\omega t \qquad \boxed{1}$$

is the equation of motion.

Now we just need to solve the differential equation.

The general solution to $\ddot{x} + \omega^2 x = 0$ is

$$x_h(t) = A\sin\omega t + B\cos\omega t$$

Since $\left(\frac{d^2}{dt^2} + \omega^2\right)\frac{F_0}{m}\sin\omega t = 0$, the particular solution is $x_p(t) = Ct\sin\omega t + Dt\cos\omega t$

So now we plug $x_p(t)$ into $\boxed{1}$ to find C and D:

$$\left.\begin{aligned} &\ddot{x}_p + \omega^2 x_p = \frac{F_0}{m}\sin\omega t \\ &2C\omega\cos\omega t - 2D\omega\sin\omega t = \frac{F_0}{m}\sin\omega t \end{aligned}\right\} \quad \text{use } \ddot{x}_p(t) = 2C\omega\cos\omega t - 2D\omega\sin\omega t - C\omega^2 t\sin\omega t - D\omega^2 t\cos\omega t$$

$$\Longrightarrow C=0, \quad D = \frac{-F_0}{2m\omega}$$

Finally, $x(t) = x_h(t) + x_p(t) = A\sin\omega t + B\cos\omega t - \frac{F_0}{2m\omega} t\cos\omega t$

From the initial conditions $x(0) = B = X_0$, $\dot{x}(0) = \omega A - \frac{F_0}{2m\omega} = V$

$$\Longrightarrow A = \frac{V}{\omega} + \frac{F_0}{2m\omega^2}.$$

$$\therefore \quad x(t) = \left[\frac{V}{\omega} + \frac{F_0}{2m\omega^2}\right]\sin\omega t + X_0\cos\omega t - \frac{F}{2m\omega}t\cos\omega t$$