The Decay $C \to A + B$:

We will apply the scattering operator to calculate the decay rate for $C \to A + B$.

Suppose that the initial state is a single C-particle with 4-momentum p_C, denoted by $|i\rangle$. The final state consists of two particles A and B with 4-momenta p_A and p_B:

$$S_{fi} = \langle p_A | \langle p_B | S | p_C \rangle = \langle p_A, p_B | S | p_C \rangle$$

S_{fi} is the transition amplitude (or "matrix element") for this decay.

S_{fi} cannot be determined analytically; it must be calculated perturbatively and we shall do so to 1st order. The perturbative expansion for the scattering operator S is given by (6.36):

$$S = 1 + \underbrace{\int_{-\infty}^{\infty} (-i H_I'(t_1)) dt_1}_{\text{first order}} + \underbrace{\int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \, (-i H_I'(t_1))(-i H_I'(t_2))}_{\text{second order}} + \cdots$$

↑ zero order

Define $A_{fi}^{(1)}$ to be S_{fi} to 1st order

$$A_{fi}^{(1)} = \langle p_A, p_B | \left[1 + \int_{-\infty}^{\infty} (-iH_I'(t_1))dt_1 \right] | p_c \rangle \qquad H' = g\int d^3x\, \phi_A \phi_B \phi_c$$

$$= g\langle p_A, p_B | \left[-i\int_{-\infty}^{\infty}\int d^3x\, \phi_A\phi_B\phi_c\, dt_1 \right] | p_c \rangle$$

$$= -ig\langle p_A, p_B | \int d^4x\, \phi_A\phi_B\phi_c | p_c \rangle$$

Before proceeding, there are two necessary digressions. First, the momentum kets must be properly normalized.

$$|p_i\rangle = \sqrt{2E_i}\; a_i^\dagger(p_i)|0\rangle \qquad \text{for } i = A, B, C$$

$$\Rightarrow \langle p_i' | p_i \rangle = 2E_i (2\pi)^3 \delta^3(\vec{p}_i' - \vec{p}_i)$$

Second, consider $\phi_c(x)|p_c\rangle$.

$$\phi_c(x)|p_c\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{1}{\sqrt{2E_k}}\left[a_c(k)e^{-ik\cdot x} + a_c^\dagger(k)e^{ik\cdot x}\right]\sqrt{2E_c}\; a_c^\dagger(p_c)|0\rangle$$

$k = (E, \vec{k})$

$$\phi_c(x)|p_c\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{\sqrt{E_c}}{\sqrt{E_k}} a_c(k)a_c^\dagger(p_c)|0\rangle + \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{\sqrt{E_c}}{\sqrt{E_k}}$$

$$\phi_c(x)|p_c\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{\sqrt{E_c}}{\sqrt{E_k}} a_c(k) a_c^\dagger(p_c) e^{-ik\cdot x}|0\rangle$$

$$+ \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{\sqrt{E_c}}{\sqrt{E_k}} a_c^\dagger(k) a_c^\dagger(p_c) e^{ik\cdot x}|0\rangle$$

In the case of $C \to A+B$ decay, the term with $a_c^\dagger(k)a_c^\dagger(p_c)$ will vanish by orthogonality b/c the final state has no C-particles.

$$\phi_c(x)|p_c\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{\sqrt{E_c}}{\sqrt{E_k}} a_c(k) a_c^\dagger(p_c) e^{-ik\cdot x}|0\rangle$$

Use $[a_i(k), a_j^\dagger(k')] = (2\pi)^3\delta^3(\vec{k}-\vec{k}')$

$$a_c(k)a_c^\dagger(p_c) = (2\pi)^3\delta^3(\vec{k}-\vec{p}_c) - a_c^\dagger(p_c)a_c(k)$$

$$\phi_c(x)|p_c\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{\sqrt{E_c}}{\sqrt{E_k}}\left[(2\pi)^3\delta^3(\vec{k}-\vec{p}_c) - a_c^\dagger(p_c)a_c(k)\right]e^{-ik\cdot x}|0\rangle$$

$$= \int d^3\vec{k}\,\frac{\sqrt{E_c}}{\sqrt{E_k}}\delta^3(\vec{k}-\vec{p}_c)e^{-ik\cdot x}|0\rangle$$

$$- \int \frac{d^3\vec{k}}{(2\pi)^3}\frac{\sqrt{E_c}}{\sqrt{E_k}} e^{-ik\cdot x} a_c^\dagger(p_c)a_c(k)|0\rangle \to 0$$

$$\phi_c(x)|p_c\rangle = e^{-ip_c\cdot x}|0\rangle$$

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where integrating over the δ-function requires $\vec{k} = \vec{p}_c$; which further implies $E_k = \sqrt{\vec{p}_k^{\,2} - m_c^2} \equiv \sqrt{\vec{p}_c^{\,2} - m_c^2} = E_c$

So the transition amplitude (to 1st order) becomes

$$A_{fi}^{(1)} = -ig\langle p_A, p_B|\int d^4x\, \phi_A\phi_B\phi_c|p_c\rangle$$

$$= -ig\int d^4x \left| e^{ip_A\cdot x} e^{ip_B\cdot x} e^{-ip_c\cdot x}\right| \langle 0|0\rangle$$

$$= -ig\int d^4x\, e^{i(p_A+p_B-p_c)\cdot x}$$

$$\boxed{A_{fi}^{(1)} = -ig(2\pi)^4\delta^{(4)}(p_A+p_B-p_c)}$$ Transition Amplitude (1st Order)

This being the amplitude, the transition probability is $|A_{fi}^{(1)}|^2$. Using the "box normalization" method of proceeding to the transition rate per unit volume, assume space has a finite volume V and the interaction occurs for a finite duration T.

$$\dot{P}_{fi} = \frac{|A_{fi}^{(1)}|^2}{VT}$$

$$\dot{P}_{fi} = (2\pi)^4 \delta^4(p_A + p_B - p_c)\,|M_{fi}|^2$$

$$\left\{ A_{fi}^{(1)} = (2\pi)^4 \delta^4(p_A + p_B - p_c)\, i M_{fi} \right\}$$
$$\left\{ i M_{fi} = -ig \right\}$$

This is the probability per unit time for transition to a particular final state f. Since a state C can decay to a continuum of final states A + B, the total decay rate will require $\dot{P}_{fi}$ to be integrated over all possible final states.

$$d\Gamma = \dot{P}_{fi}\, dN_f = \dot{P}_{fi} \frac{d^3\vec{p}_A}{(2\pi)^3 2E_A} \frac{d^3\vec{p}_B}{(2\pi)^3 2E_B}$$

$$\boxed{\Gamma = \frac{1}{2E_c}(2\pi)^4 \int \delta^4(p_A + p_B - p_c)\,|M_{fi}|^2 \frac{d^3\vec{p}_A}{(2\pi)^3 2E_A} \frac{d^3\vec{p}_B}{(2\pi)^3 2E_B}}$$

$$\Gamma = \frac{g^2}{8\pi} \frac{|\vec{p}|}{m_c^2}$$