# Two Body Decay

Suppose a particle of mass $m$ decays into two massive particles. Assume the amplitude is $M(p_2, p_3)$.

The general decay rate formula is

$$d\Gamma = \frac{1}{2m}\left(\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}\right)|M|^2 (2\pi)^4 \delta^{(4)}(p_1 - p_2 - p_3)$$

The first step is to rewrite the $\delta$-function.

$$\delta^{(4)}(p_1 - p_2 - p_3) = \delta(E_1 - E_2 - E_3)\delta^{(3)}(\vec{p}_1 - \vec{p}_2 - \vec{p}_3)$$

$$= \delta(m - E_2 - E_3)\delta^{(3)}(-\vec{p}_2 - \vec{p}_3)$$

I have assumed a center of momentum frame. Also, note that $E = \sqrt{m^2 + |\vec{p}|^2}$. The decay rate becomes

$$d\Gamma = \frac{1}{2m}\left(\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}\right)|M|^2 (2\pi)^4 \delta\left(m - \sqrt{m_2^2 + |\vec{p}_2|^2} - \sqrt{m_3^2 + |\vec{p}_3|^2}\right)\delta^{(3)}(-\vec{p}_2 - \vec{p}_3)$$

$$\Gamma = \frac{1}{8m(2\pi)^2}\int d^3p_2\, d^3p_3 \frac{|M|^2 \delta\left(m - \sqrt{m_2^2 + |\vec{p}_2|^2} - \sqrt{m_3^2 + |\vec{p}_3|^2}\right)\delta^{(3)}(-\vec{p}_2 - \vec{p}_3)}{\sqrt{m_2^2 + |\vec{p}_2|^2}\sqrt{m_3^2 + |\vec{p}_3|^2}}$$

Use $\delta^{(3)}(-\vec{p}_2 - \vec{p}_3)$ to integrate over $p_3$.

$$\Gamma = \frac{1}{8m(2\pi)^2}\int d^3p_2\, |M|^2 \frac{\delta\left(m - \sqrt{m_2^2 + |\vec{p}_2|^2} - \sqrt{m_3^2 + |\vec{p}_2|^2}\right)}{\sqrt{m_2^2 + |\vec{p}_2|^2}\sqrt{m_3^2 + |\vec{p}_2|^2}}$$

To integrate this, change variables to the total energy

$$E = \sqrt{m_2^2 + |\vec{p}_2|^2} + \sqrt{m_3^2 + |\vec{p}_2|^2}$$

and change to spherical coordinates $d^3p_2 = |\vec{p}_2|^2 d|\vec{p}_2| d\Omega$.

$$\Gamma = \frac{1}{32m\pi^2} \int d|\vec{p}_2| d\Omega |\vec{p}_2|^2 \frac{|M|^2 \delta(m-E)}{\sqrt{m_2^2+|\vec{p}_2|^2}\sqrt{m_3^2+|\vec{p}_2|^2}}$$

$$dE = \left[\frac{1}{2}\left(m_2^2+|\vec{p}_2|^2\right)^{-\frac{1}{2}}(2|\vec{p}_2|) + \frac{1}{2}\left(m_3^2+|\vec{p}_2|^2\right)^{-\frac{1}{2}}(2|\vec{p}_2|\right.$$

$$dE = \left[\frac{1}{\sqrt{m_2^2+|\vec{p}_2|^2}} + \frac{1}{\sqrt{m_3^2+|\vec{p}_2|^2}}\right] |\vec{p}_2| d|\vec{p}_2|$$

$$dE = \frac{E|\vec{p}_2| d|\vec{p}_2|}{\sqrt{m_2^2+|\vec{p}_2|^2}\sqrt{m_3^2+|\vec{p}_2|^2}}$$

$$\Gamma = \frac{1}{8m\pi} \int \frac{dE}{E} |\vec{p}_2| |M|^2 \delta(m-E)$$

$$\boxed{\Gamma = \frac{1}{8m^2\pi} |M|^2 |\vec{p}_2|}$$

where $|\vec{p}_2|$ is subject to the condition

$$E = m = \sqrt{m_2^2 + |\vec{p}_2|^2} + \sqrt{m_3^2 + |\vec{p}_2|^2}$$

$$|\vec{p}_2|^2 = \frac{1}{2m}\sqrt{m^4 + m_2^4 + m_3^4 - 2m^2m_2^2 - 2m^2m_3^2 - 2m_2^2m_3^2}$$