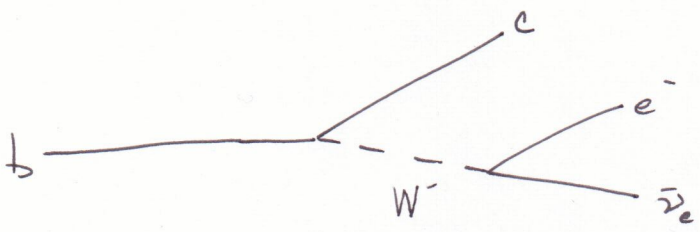
Weak Decay Rate of the b-quark

Suppose a b-quark decays to a c-quark:

$$b \to c + W^- \to c + e^- + \bar{\nu}_e$$



The decay amplitude is given to be

$$iM = -\frac{G_F V_{cb}}{\sqrt{2}} \bar{u}(c)\gamma^\alpha(1-\gamma^5)u(b)\,\bar{u}(e)\gamma_\alpha(1-\gamma^5)v_{\bar{\nu}}$$

and the decay rate is given by (4.86) Peskin-Schroeder as

$$d\Gamma = \frac{1}{2m_b}\left(\frac{d^3p(c)}{2E_c(2\pi)^3}\,\frac{d^3p(e)}{2E_e(2\pi)^3}\,\frac{d^3p(\bar{\nu})}{2E_{\bar{\nu}}(2\pi)^3}\right)|M|^2(2\pi)^4\delta^{(4)}(p_b - p_c - p_e - p_{\bar{\nu}})$$

No information is given about initial and final state spin polarization, so the probability $|M|^2$ is to be averaged over the initial spin and summed over final spins.

$$\frac{1}{2}\sum_s \sum_{r_e} \sum_{r_e} \sum_{r_\nu} \left|M(s \to r_c, r_e, r_\nu)\right|^2$$

So the current task is to square the amplitude, and to do that I need to find its complex conjugate.

$$M^* = -\frac{G_F V_{cb}}{\sqrt{2}}\underbrace{\left[\bar{u}(c)\gamma^\alpha(1-\gamma^5)u(b)\right]^*}\left(\bar{u}(e)\gamma_\alpha(1-\gamma^5)v(\bar{\nu})\right)^*$$

$$\left(\bar{u}(c)\gamma^\alpha u(b)\right)^* \cdot \left(\bar{u}(c)\gamma^\alpha\gamma^5 u(b)\right)^* = \left(u^\dagger(c)\gamma^0\gamma^\alpha u(b)\right)^* - \left(u^\dagger(c)\gamma^0\gamma^\alpha\gamma^5 u(b)\right)^*$$

$$\frac{1}{2}\sum_{\text{spins}}|M|^2$$

$$= \frac{G_F^2 V_{cb}^2}{4}\sum_{\text{spins}}\left[\bar{u}_a(c)\gamma^\alpha(1-\gamma^5)_{ab}u_b(b)\bar{u}_c(b)\gamma^\beta(1-\gamma^5)_{cd}u_d(c)\right]\left[\bar{u}_e(e)\gamma_\alpha(1-\gamma^5)_{ef}V(\bar{\nu})_f\bar{V}(\bar{\nu})_g\gamma_\beta(1-\gamma^5)_{gh}u_h(e)\right]$$

where all repeated english subscripts are implicitly summed over as a result of matrix multiplication. Now I can commute whatever I want.

$$= \frac{G_F^2 V_{cb}^2}{4}\sum_{\text{spins}}\left[u_d(c)\bar{u}_a(c)\gamma^\alpha(1-\gamma^5)_{ab}u_b(b)\bar{u}_c(b)\gamma^\beta(1-\gamma^5)_{cd}\right]\left[u_h(e)\bar{u}_e(e)\gamma_\alpha(1-\gamma^5)_{ef}V(\bar{\nu})_f\bar{V}(\bar{\nu})_g\gamma_\beta(1-\gamma^5)_{gh}\right]$$

If "s" is the spin of the charm quark then the spin sum over s is $\sum_s u(c,s)\bar{u}(c,s) = (\not{p}(c)+m_c)$, so in this case we replace "$u_d(c)\bar{u}_a(c)$" with the "da" element of $(\not{p}(c)+m_c)\to(\not{p}(c)+m_c)_{da}$. Do the same for the other three spin sums.

$$= \frac{G_F^2 V_{cb}^2}{4}\left[(\not{p}(c)+m_c)_{da}\gamma^\alpha(1-\gamma^5)_{ab}(\not{p}(b)+m_b)_{bc}\gamma^\beta(1-\gamma^5)_{cd}\right]\left[(\not{p}(e)+m_e)_{he}\gamma_\alpha(1-\gamma^5)_{ef}(\not{p}(\bar{\nu})+m_{\bar{\nu}})_{fg}\gamma_\beta(1-\gamma^5)_{gh}\right]$$

Now the matrix subscripts are cyclic, which means that as we work backwards through matrix multiplication we will come to a trace of the factors whose indices are cyclic.

$$= \frac{G_F^2 V_{cb}^2}{4}\left[\left[(\not{p}(c)+m_c)\gamma^\alpha(1-\gamma^5)\right]_{db}\left[(\not{p}(b)+m_b)\gamma^\beta(1-\gamma^5)\right]_{bd}\right]\left[\left[(\not{p}(e)+m_e)\gamma_\alpha(1-\gamma^5)\right]_{hf}\left[(\not{p}(\bar{\nu})+m_{\bar{\nu}})\gamma_\beta(1-\gamma^5)\right]_{fh}\right]$$

$$= \frac{G_F^2 V_{cb}^2}{4}\left[(\not{p}(c)+m_c)\gamma^\alpha(1-\gamma^5)(\not{p}(b)+m_b)\gamma^\beta(1-\gamma^5)\right]_{dd}\left[(\not{p}(e)+m_e)\gamma_\alpha(1-\gamma^5)(\not{p}(\bar{\nu})+m_{\bar{\nu}})\gamma_\beta(1-\gamma^5)\right]_{hh}$$

$$= \frac{G_F^2 V_{cb}^2}{4}\left[\mathrm{Tr}\left[(\not{p}(c)+m_c)\gamma^\alpha(1-\gamma^5)(\not{p}(b)+m_b)\gamma^\beta(1-\gamma^5)\right]\mathrm{Tr}\left[(\not{p}(e)+m_e)\gamma_\alpha(1-\gamma^5)(\not{p}(\bar{\nu})+m_{\bar{\nu}})\gamma_\beta(1-\gamma^5)\right]\right]$$

Despite the fact that the expression has been worked down to the product of two traces, each trace is still just one component of a rank-2 spacetime tensor given by the "$\alpha\beta$" indices.

$$= \frac{G_F^2 V_{cb}^2}{4} \mathrm{Tr}\Big[\big(\not{p}(c)\gamma^\alpha(1-\gamma^5) + m_c\gamma^\alpha(1-\gamma^5)\big)\big(\not{p}(b)\gamma^\beta(1-\gamma^5) + m_b\gamma^\beta(1-\gamma^5)\big)\Big]$$
$$\cdot \mathrm{Tr}\Big[\big(\not{p}(e)\gamma_\alpha(1-\gamma^5) + m_e\gamma_\alpha(1-\gamma^5)\big)\big(\not{p}(\bar{\nu})\gamma_\beta(1-\gamma^5) + m_{\bar{\nu}}\gamma_\beta(1-\gamma^5)\big)\Big]$$

$$= \frac{G_F^2 V_{cb}^2}{4} \mathrm{Tr}\Big[\big(\not{p}(c)\gamma^\alpha - \not{p}(c)\gamma^\alpha\gamma^5 + m_c\gamma^\alpha - m_c\gamma^\alpha\gamma^5\big)\big(\not{p}(b)\gamma^\beta - \not{p}(b)\gamma^\beta\gamma^5 + m_b\gamma^\beta - m_b\gamma^\beta\gamma^5\big)\Big]$$
$$\cdot \mathrm{Tr}\Big[\big(\not{p}(e)\gamma_\alpha - \not{p}(e)\gamma_\alpha\gamma^5 + m_e\gamma_\alpha - m_e\gamma_\alpha\gamma^5\big)\big(\not{p}(\bar{\nu})\gamma_\beta - \not{p}(\bar{\nu})\gamma_\beta\gamma^5 + m_{\bar{\nu}}\gamma_\beta - m_{\bar{\nu}}\gamma_\beta\gamma^5\big)\Big]$$

$$= \frac{G_F^2 V_{cb}^2}{4} \mathrm{Tr}\Big[\not{p}(c)\gamma^\alpha\not{p}(b)\gamma^\beta - \not{p}(c)\gamma^\alpha\not{p}(b)\gamma^\beta\gamma^5 + \not{p}(c)\gamma^\alpha\gamma^\beta m_b - \not{p}(c)\gamma^\alpha\gamma^\beta\gamma^5 m_b$$
$$- \not{p}(c)\gamma^\alpha\gamma^5\not{p}(b)\gamma^\beta + \not{p}(c)\gamma^\alpha\gamma^5\not{p}(b)\gamma^\beta\gamma^5 - \not{p}(c)\gamma^\alpha\gamma^5\gamma^\beta m_b + \not{p}(c)\gamma^\alpha\gamma^5\gamma^\beta\gamma^5 m_b$$
$$+ m_c\gamma^\alpha\not{p}(b)\gamma^\beta - m_c\gamma^\alpha\not{p}(b)\gamma^\beta\gamma^5 + m_c m_b\gamma^\alpha\gamma^5 - m_c m_b\gamma^\alpha\gamma^\beta\gamma^5$$
$$- m_c\gamma^\alpha\gamma^5\not{p}(b)\gamma^\beta + m_c\gamma^\alpha\gamma^5\not{p}(b)\gamma^\beta\gamma^5 - m_c m_b\gamma^\alpha\gamma^5\gamma^\beta + m_c m_b\gamma^\alpha\gamma^5\gamma^\beta\gamma^5\Big]$$

$$\mathrm{Tr}\Big[\not{p}(e)\gamma_\alpha\not{p}(\bar{\nu})\gamma_\beta - \not{p}(e)\gamma_\alpha\not{p}(\bar{\nu})\gamma_\beta\gamma^5 + \not{p}(e)\gamma_\alpha\gamma_\beta m_{\bar{\nu}} - \not{p}(e)\gamma_\alpha\gamma_\beta\gamma^5 m_{\bar{\nu}}$$
$$- \not{p}(e)\gamma_\alpha\gamma^5\not{p}(\bar{\nu})\gamma_\beta + \not{p}(e)\gamma_\alpha\gamma^5\not{p}(\bar{\nu})\gamma_\beta\gamma^5 - m_{\bar{\nu}}\not{p}(e)\gamma_\alpha\gamma^5\gamma_\beta + \not{p}(e)\gamma_\alpha\gamma^5\gamma_\beta\gamma^5 m_{\bar{\nu}}$$
$$+ m_e\gamma_\alpha\not{p}(\bar{\nu})\gamma_\beta - m_e\gamma_\alpha\not{p}(\bar{\nu})\gamma_\beta\gamma^5 + m_e m_{\bar{\nu}}\gamma_\alpha\gamma_\beta - m_e m_{\bar{\nu}}\gamma_\alpha\gamma_\beta\gamma^5$$
$$- m_e\gamma_\alpha\gamma^5\not{p}(\bar{\nu})\gamma_\beta + m_e\gamma_\alpha\gamma^5\not{p}(\bar{\nu})\gamma_\beta\gamma^5 - m_e m_{\bar{\nu}}\gamma_\alpha\gamma^5\gamma_\beta + m_e m_{\bar{\nu}}\gamma_\alpha\gamma^5\gamma_\beta\gamma^5\Big]$$

where I have used $\mathrm{Tr}[\text{odd \# of } \gamma_i] = 0$ as well as the following:

(i) $\mathrm{Tr}\left[\not{p}(c)\gamma^\alpha\gamma^5\gamma^\beta\right] = -\mathrm{Tr}\left[\gamma^\rho \not{p}_\rho(c)\gamma^\alpha\gamma^\beta\gamma^5\right]$

$= -\mathrm{Tr}\left[\not{p}_\rho(c)\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^5\right]$

$= -\frac{1}{2}\left[\mathrm{Tr}\left(\not{p}_\rho\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^5\right) + \mathrm{Tr}\left(\not{p}_\rho\gamma^\rho\gamma^\alpha\gamma^\beta\gamma^5\right)\right]$

$= -\frac{1}{2}\mathrm{Tr}\left(\not{p}_\rho\gamma^\rho\{\gamma^\alpha,\gamma^\beta\}\gamma^5\right)$

$= -\frac{1}{2}\mathrm{Tr}\left(\not{p}_\rho\gamma^\rho 2g^{\alpha\beta}\gamma^5\right)$

$= -g^{\alpha\beta}\mathrm{Tr}\left(\not{p}_\rho\gamma^\rho\gamma^5\right)$

$= 0$

(ii) $\mathrm{Tr}\left(\not{p}(c)\gamma^\alpha\gamma^5\not{p}(b)\gamma^\beta\gamma^5\right) = (-1)^2\mathrm{Tr}\left(\not{p}(c)\gamma^\alpha\not{p}(b)\gamma^\beta\gamma^5\gamma^5\right)$

$= \mathrm{Tr}\left(\not{p}(c)\gamma^\alpha\not{p}(b)\gamma^\beta\right)$

(iii) $\mathrm{Tr}\left(m_c\gamma^\alpha\gamma^5\not{p}(b)\gamma^\beta\gamma^5\right) = (-1)^2 m_c\mathrm{Tr}\left(\gamma^\alpha\not{p}(b)\gamma^\beta\gamma^5\gamma^5\right)$

$= m_c\mathrm{Tr}\left(\gamma^\alpha\not{p}_\mu(b)\gamma^\mu\gamma^\beta\right)$

$= 0$

Back to the squared-amplitude.

$= \frac{G_F^2 V_{cb}}{4}\mathrm{Tr}\left[\not{p}(c)\gamma^\alpha\not{p}(b)\gamma^\beta - \not{p}(c)\gamma^\alpha\not{p}(b)\gamma^\beta\gamma^5 - \not{p}(c)\gamma^\alpha\not{p}(b)\gamma^\beta\gamma^5 + \not{p}(c)\gamma^\alpha\not{p}(b)\gamma^\beta + \cancel{m_c m_b \gamma^\alpha\gamma^\beta} - \cancel{m_c m_b \gamma^\alpha\gamma^\beta}\right]$

$\mathrm{Tr}\left[\not{p}(e)\gamma_\alpha\not{p}(\bar{\nu})\gamma_\beta - \not{p}(e)\gamma_\alpha\not{p}(\bar{\nu})\gamma_\beta\gamma^5 - \not{p}(e)\gamma_\alpha\not{p}(\bar{\nu})\gamma_\beta\gamma^5 + \not{p}(e)\gamma_\alpha\not{p}(\bar{\nu})\gamma_\beta + \cancel{m_e m_{\bar{\nu}}\gamma_\alpha\gamma_\beta} - \cancel{m_e m_{\bar{\nu}}\gamma_\alpha\gamma_\beta}\right]$

$$= \frac{G_F^2 V_{cb}^2}{4} \mathrm{Tr}\left[2\not{p}(c)\gamma^\alpha \not{p}(b)\gamma^\beta - 2\not{p}(c)\gamma^\alpha \not{p}(b)\gamma^\beta\gamma^5\right] \mathrm{Tr}\left[2\not{p}(e)\gamma_\alpha \not{p}(\bar{\nu})\gamma_\beta - 2\not{p}(e)\gamma_\alpha \not{p}(\bar{\nu})\gamma_\beta\gamma^5\right]$$

$$= G_F^2 V_{cb}^2 \mathrm{Tr}\left[\not{p}(c)\gamma^\alpha \not{p}(b)\gamma^\beta - \not{p}(c)\gamma^\alpha \not{p}(b)\gamma^\beta\gamma^5\right] \mathrm{Tr}\left[\not{p}(e)\gamma_\alpha \not{p}(\bar{\nu})\gamma_\beta - \not{p}(e)\gamma_\alpha \not{p}(\bar{\nu})\gamma_\beta\gamma^5\right]$$

Let's individually calculate these last couple traces.

$$\mathrm{Tr}\left[\not{p}(c)\gamma^\alpha \not{p}(b)\gamma^\beta\right]$$

$$= \mathrm{Tr}\left[p_\mu(c)\gamma^\mu\gamma^\alpha p_\nu(b)\gamma^\nu\gamma^\beta\right]$$

$$= p_\mu(c)p_\nu(b)\,\mathrm{Tr}\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\right]$$

$$= 4\,p_\mu(c)p_\nu(b)\left[g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\nu}\right]$$

$$= 4\left[p^\alpha(c)p^\beta(b) - p^\nu(c)p_\nu(b)g^{\alpha\beta} + p^\beta(c)p^\alpha(b)\right]$$

$$= 4\left[p^\alpha(c)p^\beta(b) + p^\beta(c)p^\alpha(b) - p(c)\cdot p(b)\,g^{\alpha\beta}\right]$$

$$\mathrm{Tr}\left[\not{p}(c)\gamma^\alpha \not{p}(b)\gamma^\beta\gamma^5\right]$$

$$= \mathrm{Tr}\left[p_\mu(c)\gamma^\mu\gamma^\alpha p_\nu(b)\gamma^\nu\gamma^\beta\gamma^5\right]$$

$$= p_\mu(c)p_\nu(b)\,\mathrm{Tr}\left[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^5\right]$$

$$= -4i\,p_\mu(c)p_\nu(b)\,\varepsilon^{\mu\alpha\nu\beta}$$

Where $\varepsilon = 0$, unless $\mu\alpha\nu\beta = 0123$ or some permutation of 0123.

Back to the squared amplitude again,

$$= 16 G_F^2 V_{cb}^2 \left[p^\alpha(c)p^\beta(b) + p^\beta(c)p^\alpha(b) - p(c)\cdot p(b)\,g^{\alpha\beta} - i\,p_\mu(c)p_\nu(b)\,\varepsilon^{\mu\alpha\nu\beta}\right]$$

$$\left[p_\alpha(e)p_\beta(\bar{\nu}) + p_\beta(e)p_\alpha(\bar{\nu}) - p(e)\cdot p(\bar{\nu})\,g_{\alpha\beta} - i\,p^\rho(e)\,p^\sigma(\bar{\nu})\,\varepsilon_{\rho\sigma\alpha\beta}\right]$$

$$= (\gamma^\alpha u(b))^* (u^\dagger(c)\gamma^0)^* - (\gamma^\alpha\gamma^5 u(b))^* (u^\dagger(c)\gamma^0)^*$$

$$= u^{T*}(b)(\gamma^\alpha)^{T*}(\gamma^0)^{T*}(u^\dagger(c))^{T*} - u^{T*}(b)(\gamma^\alpha\gamma^5)^{T*}(\gamma^0)^{T*}(u^\dagger(c))^{T*}$$

$$= u^\dagger(b)(\gamma^\alpha)^\dagger(\gamma^0)^\dagger u(c) - u^\dagger(b)(\gamma^\alpha\gamma^5)^\dagger(\gamma^0)^\dagger u(c)$$

$$= u^\dagger(b)(\gamma^\alpha)^\dagger\gamma^0 u(c) - u^\dagger(b)(\gamma^5)^\dagger(\gamma^\alpha)^\dagger\gamma^0 u(c)$$

$$= u^\dagger(b)\gamma^0\gamma^\alpha u(c) - u^\dagger(b)\gamma^0\gamma^\alpha\gamma^5 u(c)$$

$$= \bar{u}(b)\gamma^\alpha u(c) - \bar{u}(b)\gamma^\alpha\gamma^5 u(c)$$

$$= \bar{u}(b)\gamma^\alpha(1-\gamma^5)u(c)$$

By a similar argument (just switching the indices) the second factor in M^* becomes

$$(\bar{u}(e)\gamma_\alpha(1-\gamma^5)v_{\bar{\nu}})^* = \bar{v}_{\bar{\nu}}\gamma_\alpha(1-\gamma^5)u(e)$$

$$\Rightarrow M^* = -\frac{G_F V_{cb}}{\sqrt{2}}\left(\bar{u}(b)\gamma^\alpha(1-\gamma^5)u(c)\right)\left(\bar{v}(\bar{\nu})\gamma_\alpha(1-\gamma^5)u(e)\right)$$

$$\Rightarrow |M|^2 = \frac{G_F^2 V_{cb}^2}{2}\left[\bar{u}(c)\gamma^\beta(1-\gamma^5)u(b)\bar{u}(b)\gamma^\beta(1-\gamma^5)u(c)\right]\left[\bar{u}(e)\gamma_\alpha(1-\gamma^5)v(\bar{\nu})\bar{v}(\bar{\nu})\gamma_\beta(1-\gamma^5)u(e)\right]$$

Next, I want to compute $\frac{1}{2}\sum_{\text{spins}}|M|^2$. If $u(c)$ can be commuted to the left of $\bar{u}(c)$ we can use the spin sums to simplify the expression. Notice how, for example, $\bar{u}(c)\gamma^\alpha(1-\gamma^5)u(b)$ is a row vector times a 4×4 matrix times a column vector. That means this term is itself a scalar. This reveals how to rewrite the same line in terms of individual matrix elements rather than matrices.

$$= 16 G_F^2 V_{cb}^2 \Big[p^\alpha(c)\, p^\beta(b)\, p_\alpha(e)\, p_\beta(\bar\nu) + p^\alpha(c)\, p^\beta(b)\, p_\beta(e)\, p_\alpha(\bar\nu)$$

$$- p^\alpha(c)\, p^\beta(b)\, p(e)\cdot p(\bar\nu)\, g_{\alpha\beta} - i p^\tau(c)\, p^\beta(b)\, p_\rho(e)\, p_\sigma(\bar\nu)\, \varepsilon^{\rho\ \sigma}_{\tau\ \beta}$$

$$+ p^\beta(c)\, p^\alpha(b)\, p_\alpha(e)\, p_\beta(\bar\nu) + p^\beta(c)\, p^\alpha(b)\, p_\beta(e)\, p_\alpha(\bar\nu)$$

$$- p^\beta(c)\, p^\alpha(b)\, p(e)\cdot p(\bar\nu)\, g_{\alpha\beta} - i p^\beta(c)\, p^\alpha(b)\, p_\rho(e)\, p_\sigma(\bar\nu)\, \varepsilon^{\rho\ \sigma}_{\alpha\ \beta}$$

$$- p(c)\cdot p(b)\, g^{\alpha\beta} p_\alpha(e)\, p_\beta(\bar\nu) - p(c)\cdot p(b)\, g^{\alpha\beta} p_\beta(e)\, p_\alpha(\bar\nu)$$

$$+ p(c)\cdot p(b)\, g^{\alpha\beta} p(e)\cdot p(\bar\nu)\, g_{\alpha\beta} + i p(c)\cdot p(b)\, p_\rho(e)\, p_\sigma(\bar\nu)\, \varepsilon^{\rho\ \sigma}_{\alpha\ \beta} g^{\alpha\beta} \to 0$$

$$- i p_\mu(c)\, p_\nu(b)\, p_\alpha(e)\, p_\beta(\bar\nu)\, \varepsilon^{\mu\alpha\nu\beta} - i p_\mu(c)\, p_\nu(b)\, p_\beta(e)\, p_\alpha(\bar\nu)\, \varepsilon^{\mu\alpha\nu\beta}$$

$$+ i p_\mu(c)\, p_\nu(b)\, p(e)\cdot p(\bar\nu)\, g_{\alpha\beta}\, \varepsilon^{\mu\alpha\nu\beta} \to 0 + i^2 p_\mu(c)\, p_\nu(b)\, p_\beta(e)\, p_\sigma(\bar\nu)\, \varepsilon^{\mu\alpha\nu\beta} \varepsilon^{\rho\ \sigma}_{\alpha\ \beta} \Big]$$

Some of these terms are repeated and combine. Two vanish b/c $g_{\alpha\beta}\varepsilon^{\mu\alpha\nu\beta} = \varepsilon^{\mu\ \nu\beta}_{\ \beta} = 0$ (repeated "β" index means it vanishes).

The last term vanishes unless $\{\mu=\rho \text{ and } \nu=\sigma\}$ or vice versa.

Let's calculate it now:

$$i^2 p_\rho(c)\, p_\nu(b)\, p_\rho(e)\, p_\sigma(\bar\nu)\, \varepsilon^{\mu\alpha\nu\beta} \varepsilon^{\rho\ \sigma}_{\alpha\ \beta}$$

$$= 2\, p_\mu(c)\, p_\nu(b)\, p_\rho(e)\, p_\sigma(\bar\nu) \left[\delta^{\mu\rho}\delta^{\nu\sigma} - \delta^{\mu\sigma}\delta^{\nu\rho} \right]$$

$$= 2 \left[p^\rho(c)\, p^\sigma(b)\, p_\rho(e)\, p_\sigma(\bar\nu) - p^\sigma(c)\, p^\rho(b)\, p_\rho(e)\, p_\sigma(\bar\nu) \right]$$

$$= 2 \left[(p(c)\cdot p(e))(p(b)\cdot p(\bar\nu)) - ((p(c)\cdot p(\bar\nu))(p(b)\cdot p(e)) \right]$$

Now finish the amplitude.

$$= 16 G_F^2 V_{cb}^2 \Big[(p(c)\cdot p(e))(p(b)\cdot p(\bar{\nu})) + (p(c)\cdot p(\bar{\nu}))(p(b)\cdot p(e)) - (p(c)\cdot p(b))(p(e)\cdot p(\bar{\nu})) - i p^{\alpha}(c) p^{\beta}(b) p_{\rho}(e) p_{\sigma}(\bar{\nu}) \varepsilon^{\rho\sigma}{}_{\alpha\beta}$$

$$+ (p(c)\cdot p(\bar{\nu}))(p(b)\cdot p(e)) + (p(c)\cdot p(e))(p(\bar{\nu})\cdot p(b)) - (p(c)\cdot p(b))(p(e)\cdot p(\bar{\nu})) - i p^{\beta}(c) p^{\alpha}(b) p_{\rho}(e) p_{\sigma}(\bar{\nu}) \varepsilon^{\rho\sigma}{}_{\alpha\beta}$$

$$- (p(c)\cdot p(b))(p(e)\cdot p(\bar{\nu})) - (p(c)\cdot p(b))(p(e)\cdot p(\bar{\nu})) + 4(p(c)\cdot p(b))(p(e)\cdot p(\bar{\nu}))$$

$$- i p_{\mu}(c) p_{\nu}(b) p_{\alpha}(e) p_{\beta}(\bar{\nu}) \varepsilon^{\mu\alpha\nu\beta} - i p_{\mu}(c) p_{\nu}(b) p_{\beta}(e) p_{\alpha}(\bar{\nu}) \varepsilon^{\mu\alpha\nu\beta} + 2(p(c)\cdot p(e))(p(b)\cdot p(\bar{\nu})) - 2(p(c)\cdot p(\bar{\nu}))(p(b)\cdot p(e)) \Big]$$

Now, since the γ-matrices anticommute, interchanging any two of the greek indices in $\mathrm{Tr}[\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^{5}] = -4i\varepsilon^{\mu\nu\rho\sigma}$ will just change its sign. Because of this, the remaining four terms with a Levi-Civita ε factor will cancel each other.

$$= 16 G_F^2 V_{cb}^2 \Big[2(p(c)\cdot p(e))(p(b)\cdot p(\bar{\nu})) + 2(p(c)\cdot p(\bar{\nu}))(p(b)\cdot p(e)) + 2(p(c)\cdot p(e))(p(b)\cdot p(\bar{\nu})) - 2(p(c)\cdot p(\bar{\nu}))(p(b)\cdot p(e)) \Big]$$

$$= 64 G_F^2 V_{cb}^2 (p(c)\cdot p(e))(p(b)\cdot p(\bar{\nu})) = \frac{1}{2}\sum_{\text{spins}} |M|^2$$

Finally, we can get back to the decay rate.

$$d\Gamma = \frac{1}{2m_b}\left(\frac{d^3p(c)}{2E_c(2\pi)^3}\,\frac{d^3p(e)}{2E_e(2\pi)^3}\,\frac{d^3p(\bar{\nu})}{2E_{\bar{\nu}}(2\pi)^3}\right)\left(\frac{1}{2}\sum_{\text{spins}}|M|^2\right)(2\pi)^4\delta^{(4)}(p_b - p_c - p_e - p_{\bar{\nu}})$$

See "Muon Decay" for the rest