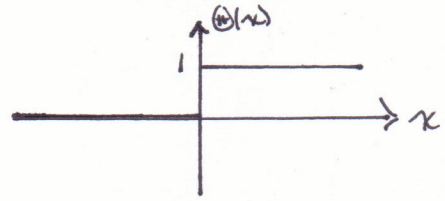
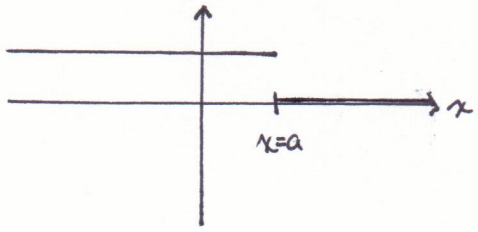
6.5 Calculation of a Feynman Propagator in ABC Theory

First, observe some usages of the Heaviside Step Function.

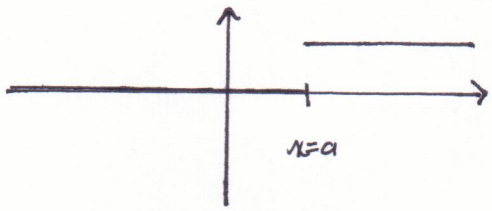
$\Theta(x)$:



$\Theta(x-a)$:



$\Theta(a-x)$:



The Feynman Propagator to be calculated is

$$\langle 0|T(\phi_c(x_1)\phi_c(x_2))|0\rangle = \begin{cases} \langle 0|\phi_c(x_1)\phi_c(x_2)|0\rangle & \text{for } t_1 > t_2 \\ \langle 0|\phi_c(x_2)\phi_c(x_1)|0\rangle & \text{for } t_2 > t_1 \end{cases}$$

Each of the ϕ_c's can create or destroy C-quanta (look at the mode expansion for ϕ_c), but for the v.e.v. to be non zero anything created in the initial state must be destroyed in the final state.

If $t_1 > t_2$, a C-quanta is created at x_2 (the earlier coordinate) and destroyed at x_1 (the later coordinate). If $t_2 > t_1$, vice-versa. It is called a "propagator" b/c it describes a quantum propagating b/w two spacetime points. Also notice how causality is maintained by the time-ordering operator to ensure a particle is created before being destroyed, even if it is virtual.

$$= \langle 0 | \Big[\underbrace{\Theta(t_1 - t_2)\, \phi_c(x_1)\, \phi_c(x_2)}_{\text{for } t_1 > t_2} + \underbrace{\Theta(t_2 - t_1)\, \phi_c(x_2)\, \phi_c(x_1)}_{\text{for } t_2 > t_1} \Big] | 0 \rangle$$

$$= \langle 0 | \Theta(t_1 - t_2) \int \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega}} \left[a(\vec{k}) e^{-i\vec{k}\cdot x_1} + a^\dagger(\vec{k}) e^{i\vec{k}\cdot x_1} \right] \int \frac{d^3\vec{k}'}{(2\pi)^3 \sqrt{2\omega'}} \left[a(\vec{k}') e^{-i\vec{k}'\cdot x_2} + a^\dagger(\vec{k}') e^{i\vec{k}'\cdot x_2} \right] | 0 \rangle$$

$$+ \langle 0 | \Theta(t_2 - t_1) \int \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega}} \left[a(\vec{k}) e^{-i\vec{k}\cdot x_2} + a^\dagger(\vec{k}) e^{i\vec{k}\cdot x_2} \right] \int \frac{d^3\vec{k}'}{(2\pi)^3 \sqrt{2\omega'}} \left[a(\vec{k}') e^{-i\vec{k}'\cdot x_1} + a^\dagger(\vec{k}') e^{i\vec{k}'\cdot x_1} \right] | 0 \rangle$$

$$= \langle 0 | \Theta(t_1 - t_2) \int\!\!\int \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^6\, 2\sqrt{\omega\omega'}} \Big[a(\vec{k}) a(\vec{k}') e^{-i\vec{k}\cdot x_1} \cancel{e^{-i\vec{k}'\cdot x_2}}^{\,0} + a(\vec{k}) a^\dagger(\vec{k}') e^{-i\vec{k}\cdot x_1} e^{i\vec{k}'\cdot x_2}$$
$$+ \cancel{a^\dagger(\vec{k}) a(\vec{k}') e^{i\vec{k}\cdot x_1} e^{-i\vec{k}'\cdot x_2}}^{\,0} + \cancel{a^\dagger(\vec{k}) a^\dagger(\vec{k}') e^{i\vec{k}\cdot x_1} e^{i\vec{k}'\cdot x_2}}^{\,0} \Big] | 0 \rangle$$

$$+ \langle 0 | \Theta(t_2 - t_1) \int\!\!\int \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^6\, 2\sqrt{\omega\omega'}} \Big[\cancel{a(\vec{k}) a(\vec{k}') e^{-i\vec{k}\cdot x_2} e^{-i\vec{k}'\cdot x_1}}^{\,0} + a(\vec{k}) a^\dagger(\vec{k}') e^{-i\vec{k}\cdot x_2} e^{i\vec{k}'\cdot x_1}$$
$$+ \cancel{a^\dagger(\vec{k}) a(\vec{k}') e^{i\vec{k}\cdot x_2} e^{-i\vec{k}'\cdot x_1}}^{\,0} + \cancel{a^\dagger(\vec{k}) a^\dagger(\vec{k}') e^{i\vec{k}\cdot x_2} e^{i\vec{k}'\cdot x_1}}^{\,0} \Big] | 0 \rangle$$

where I used $a(\vec{k}) | 0 \rangle = 0 = \langle 0 | a^\dagger(\vec{k}')$.

$$= \langle 0 | \Theta(t_1 - t_2) \iint \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^6 2\sqrt{\omega\omega'}} a(k) a^\dagger(k') e^{-ik\cdot x_1} e^{ik'\cdot x_2} | 0 \rangle$$

$$+ \langle 0 | \Theta(t_2 - t_1) \iint \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^6 2\sqrt{\omega\omega'}} a(k) a^\dagger(k') e^{-ik\cdot x_2} e^{ik'\cdot x_1} | 0 \rangle$$

Now use the commutation relation

$$[a(k), a^\dagger(k')] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}')$$

$$\Rightarrow a(k) a^\dagger(k') = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') + a^\dagger(k') a(k)$$

$$\langle 0 | T(\phi_c(x_1) \phi_c(x_2)) | 0 \rangle$$

$$= \langle 0 | \Theta(t_1 - t_2) \iint \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^6 2\sqrt{\omega\omega'}} (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') e^{-ik\cdot x_1} e^{ik'\cdot x_2} | 0 \rangle$$

$$+ \langle 0 | \Theta(t_2 - t_1) \iint \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^6 2\sqrt{\omega\omega'}} (2\pi)^3 \delta^3(\vec{k} - \vec{k}') e^{-ik\cdot x_2} e^{ik'\cdot x_1} | 0 \rangle$$

$$= \langle 0 | \Theta(t_1 - t_2) \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega} e^{ik\cdot(x_1 - x_2)} | 0 \rangle$$

$$+ \langle 0 | \Theta(t_2 - t_1) \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega} e^{ik\cdot(x_2 - x_1)} | 0 \rangle$$

$$= \langle 0 | 0 \rangle \Theta(t_1 - t_2) \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega} e^{ik\cdot(x_1 - x_2)} + \langle 0 | 0 \rangle \Theta(t_2 - t_1) \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega} e^{ik\cdot(x_2 - x_1)}$$

$$= \boxed{\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega} \left[\Theta(t_1 - t_2) e^{ik\cdot(x_1 - x_2)} + \Theta(t_2 - t_1) e^{ik\cdot(x_2 - x_1)} \right]}$$

Change the propagator into a manifestly invariant form.

Express the step function as

$$\Theta(t) = i\int_{-\infty}^{\infty} \frac{dz'}{2\pi} \frac{e^{-iz't}}{z'+i\varepsilon} \qquad \text{where } \varepsilon \to 0$$

Let $z' = z - \omega$, so $dz' = dz$, and multiply by $e^{-i\omega t}$

$$\Theta(t)\, e^{-i\omega t} = i\int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{-i(z-\omega)t}\, e^{-i\omega t}}{(z-\omega)+i\varepsilon}$$

$$\Theta(t)\, e^{-i\omega t} = i\int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{-izt}}{z-(\omega-i\varepsilon)}$$

Use this in the propagator where $t \to t_1 - t_2$ or $t \to t_2 - t_1$.

$$\langle 0|T|\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega}\left[\Theta(t_1-t_2)e^{ik(x_1-x_2)} + \Theta(t_2-t_1)e^{ik(x_2-x_1)}\right]$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega}\left[\Theta(t_1-t_2)e^{-i\omega(t_1-t_2)+i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)} + \Theta(t_2-t_1)e^{-i\omega(t_2-t_1)+i\vec{k}\cdot(\vec{x}_2-\vec{x}_1)}\right]$$

$$= \int \frac{d^3\vec{k}\, dz}{(2\pi)^3 2\omega}\left[\frac{i}{2\pi}\frac{e^{-iz(t_1-t_2)}\, e^{i\vec{k}\cdot(\vec{x}_1-\vec{x}_2)}}{z-(\omega-i\varepsilon)} + \frac{i}{2\pi}\frac{e^{-iz(t_2-t_1)}\, e^{i\vec{k}\cdot(\vec{x}_2-\vec{x}_1)}}{z-(\omega-i\varepsilon)}\right]$$

Let's change notation so that $z \equiv k_0$; however, we do <u>not</u> have $k_0 = \sqrt{\vec{k}^2 + m_c^2}$. Thus, $k_0 \neq \omega_k$, where ω_k is also thought of as a zeroth component of the 4-vector k.

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i}{2\omega} \left[\frac{e^{-ik\cdot(x_1-x_2)}}{k_0 - (\omega - i\varepsilon)} + \frac{e^{ik\cdot(x_2-x_1)}}{k_0 - (\omega - i\varepsilon)} \right]$$

Now for the bizarre step: change $k \to -k$ in the 2nd term only. For this move to be valid, the must be a symmetry about the origin so that

$$\frac{e^{ik\cdot(x_2-x_1)}}{k_0 - (\omega - i\varepsilon)} = \frac{e^{i(-k)\cdot(x_2-x_1)}}{(-k_0) - (\omega - i\varepsilon)}$$

$$\Rightarrow \langle 0 | T(\phi(x_1)\phi(x_2)) | 0 \rangle$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x_1-x_2)} \cdot \frac{i}{2\omega} \left[\frac{1}{k_0 - (\omega_k - i\varepsilon)} - \frac{1}{k_0 + \omega - i\varepsilon} \right]$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x_1-x_2)} \frac{i}{k_0^2 - (\omega_k - i\varepsilon)^2}$$

$$= \boxed{\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x_1-x_2)} \frac{i}{k_0^2 - \vec{k}^2 - m_c^2 + i\varepsilon}}$$