$A+B \rightarrow A+B$ Scattering

We are calculating the scattering matrix element S_{fi} for $A+B \rightarrow A+B$. The calculation is approached perturbatively, so to 2nd order we have the following:

$$S_{fi} = \langle p_A' | \langle p_B' | S | p_A \rangle | p_B \rangle$$

$$S_{fi} = \langle p_A', p_B' | S | p_A, p_B \rangle$$

Zero-Order Contribution:

$$S_{fi}^{(0)} = \langle p_A', p_B' | p_A, p_B \rangle$$

$$= \langle 0 | \sqrt{2E_A'}\, a_A(p_A') \sqrt{2E_B'}\, a_B(p_B') \sqrt{2E_A}\, a_A^\dagger(p_A) \sqrt{2E_B}\, a_B^\dagger(p_B) | 0 \rangle$$

$$= \langle 0 | a_A(p_A') a_B(p_B') a_A^\dagger(p_A) a_B^\dagger(p_B) | 0 \rangle \sqrt{16 E_A' E_B' E_A E_B}$$

All operators producing A particles commute with operators producing B particles.

$$S_{fi}^{(0)} = \langle 0 | a_A(p_A') a_A^\dagger(p_A) a_B(p_B') a_B^\dagger(p_B) | 0 \rangle \sqrt{16 E_A' E_B' E_A E_B}$$

Now use the commutation relation

$$[a_i(\vec{p}), a_i^\dagger(\vec{p}')] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \quad \text{for } i = A, B \qquad (6.46)$$

$$a_i(p_i') a_i^\dagger(p_i) = (2\pi)^3 \delta^3(\vec{p}_i' - \vec{p}_i) + a_i^\dagger(p_i) a_i(p_i')$$

$$\Rightarrow S_{fi}^{(0)} = \langle 0 | \left[(2\pi)^3 \delta^3(\vec{p}_A' - \vec{p}_A) + a_A^\dagger(p_A) a_A(p_A') \right] a_B(p_B') a_B^\dagger(p_B) | 0 \rangle \sqrt{16 E_A' E_B' E_A E_B}$$

$$= \langle 0 | (2\pi)^3 \delta^3(\vec{p}_A' - \vec{p}_A) a_B(p_B') a_B^\dagger(p_B) | 0 \rangle \sqrt{16 E_A' E_B' E_A E_B}$$

$$= \langle 0 | (2\pi)^3 \delta^3(\vec{p}_A' - \vec{p}_A) \left[(2\pi)^3 \delta^3(\vec{p}_B' - \vec{p}_B) + a_B^\dagger(p_B) a_B(p_B') \right] | 0 \rangle \sqrt{16 E_A' E_B' E_A E_B}$$

$$= (2\pi)^3 \delta^3(\vec{p}_A' - \vec{p}_A) (2\pi)^3 \delta^3(\vec{p}_B' - \vec{p}_B) \sqrt{16 E_A' E_B' E_A E_B}$$

$$= 2E_A (2\pi)^3 \delta^3(\vec{p}_A' - \vec{p}_A) \cdot 2E_B (2\pi)^3 \delta^3(\vec{p}_B' - \vec{p}_B)$$

$$= \langle p_A' | p_A \rangle \langle p_B' | p_B \rangle$$

$$= 1.$$

Where the last step assumes the states are properly normalized. This contribution represents propagation with no interaction, illustrated by the absence of g, ($g^0 = 1$).

The 1st order contribution

$$S_{fi}^{(1)} = -ig \int d^4x \, \langle p_A' p_B' | \phi_A(x) \phi_B(x) \phi_C(x) | p_A p_B \rangle$$

vanishes as there is no C-particle in either the final or initial state. $S_{fi}^{(1)} = 0$.

Thus, the 2nd order contribution is the first term that expresses an interaction between particles.

$$S_{fi}^{(2)} = \frac{(-ig)^2}{2} \iint d^4x_1 \, d^4x_2 \, \langle p_A' p_B' | T\left[\mathcal{H}_1(x_1) \mathcal{H}_2(x_2)\right] | p_A p_B \rangle$$

$$= \frac{(-ig)^2}{2} \iint d^4x_1 \, d^4x_2 \, \langle 0 | a_A(p_A') a_B(p_B') T\left[\phi_A(x_1) \phi_B(x_1) \phi_C(x_1) \right.$$

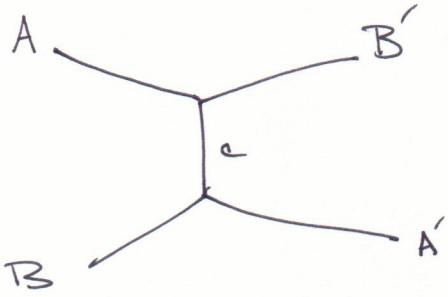
$$\left. \phi_A(x_2) \phi_B(x_2) \phi_C(x_2)\right] a_A^\dagger(p_A) a_B^\dagger(p_B) | 0 \rangle \sqrt{16 E_A' E_B' E_A E_B}$$

This beast has 10 operators in the vacuum expectation value. We need to apply Wick's Theorem to simplify the v.e.v. As a quick digressio suppose A, B, C, D are operators. Then by Wick's Theorem

$$\langle 0|ABCD|0\rangle = \langle 0|AB|0\rangle\langle 0|CD|0\rangle + \langle 0|AC|0\rangle\langle 0|BD|0\rangle + \langle 0|AD|0\rangle\langle 0|BC|0$$

Applying this concept to $S_{fi}^{(2)}$, we find all nonvanishing terms listed on page 159 and they all contain the Feynman propagator $\langle 0|T[\phi_c(x_1)\phi_c(x_2)]|0\rangle$.

Let's focus on the term of (6.89).



It was found in problem 6.5 that this term is

$$(-ig)^2 \iint d^4x_1\, d^4x_2\, e^{i(p_A'-p_B)\cdot x_1}\, e^{i(p_B'-p_A)\cdot x_2}\, \langle 0|T[\phi_c(x_1)\phi_c(x_2)]|0\rangle$$

and the propagator was calculated to be

$$\langle 0|T[\phi_c(x_1)\phi_c(x_2)]|0\rangle = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot(x_1-x_2)}\, \frac{i}{k_0^2-\vec{k}^2-m_c^2+i\varepsilon} \equiv f(x_1-x_2)$$

(6.6) Introduce $x \equiv x_1 - x_2$ and $X \equiv \frac{x_1+x_2}{2}$.

$$= \iint d^4x\, d^4X\, e^{i(p_A'-p_B)\cdot(X+\frac{x}{2})}\, e^{i(p_B'-p_A)\cdot(X-\frac{x}{2})}\, f(x)$$

Integrate over X using (E.26).

$$= (-ig)^2 (2\pi)^4 \delta^4(p_A' - p_B + p_B' - p_A) \int d^4x \, e^{i[(p_A' - p_B) - (p_B' - p_A)] \cdot \frac{x}{2}} f(x)$$

Notice that the δ-function enforces

$$(p_A' - p_B) = -(p_B' - p_A) \equiv q$$

As such,

$$= (-ig)^2 (2\pi)^4 \delta^4(p_A' + p_B' - p_A - p_B) \int d^4x \, e^{-i(2q)\cdot \frac{x}{2}} f(x)$$

$$= (-ig)^2 (2\pi)^4 \delta^4(p_A' + p_B' - p_A - p_B) \int d^4x \, e^{-iq\cdot x} f(x)$$

$$= \boxed{(-ig)^2 (2\pi)^4 \delta^4(p_A' + p_B' - p_A - p_B) \frac{i}{q^2 - m_c^2 + i\varepsilon}}$$

Similar to the $C \to A + B$ decay (6.56), the transition rate per unit volume is

$$\dot{P}_{fi} = (2\pi)^4 \delta^4(p_A + p_B - p_A' - p_B') \left| M_{fi} \right|^2$$

And we finally get to the cross section

$$\sigma = \int d\sigma$$

$$\sigma = \frac{1}{2E_B 2E_A |\vec{v}_A|} \int \dot{P}_{fi} \, dN_f$$

where $\vec{v}_A$ is the velocity of the incident A particle and we are in the rest frame of the initial B particle.

$$\sigma = \frac{1}{2E_B 2E_A |\vec{v}_A|} (2\pi)^4 \int \delta^4(p_A + p_B - p_A' - p_B') |M_{fi}|^2 \frac{d^3\vec{p}_A'}{(2\pi)^3 2E_A'} \frac{d^3\vec{p}_B'}{(2\pi)^3 2E_B'}$$

$$\sigma = \frac{1}{4E_A E_B |\vec{v}_A|} \int |M_{fi}|^2 \, dLips(s; p_A', p_B')$$

$$dLips(s; p_A', p_B') = \frac{1}{(4\pi)^2} \delta^4(p_A + p_B - p_A' - p_B') \frac{d^3\vec{p}_A'}{E_A'} \frac{d^3\vec{p}_B'}{E_B'}$$

Let's look at the Lorentz Invariant Phase Space.

$$\int d(P.S._2) = \frac{1}{(4\pi)^2} \int \frac{d^3\vec{p}_A'}{E_A'} \int \frac{d^3\vec{p}_B'}{E_B'} \delta^4(p_A + p_B - p_A' - p_B')$$

Integrate over $d^3\vec{p}_B'$.

$$\int d(P.S._2) = \frac{1}{(4\pi)^2} \int \frac{d^3\vec{p}_A'}{E_A'} \frac{1}{E_B'} \delta(E_A + E_B - E_A' - E_B')$$

The δ-function gave us the constraint $\vec{p}_B' = \vec{p}_A + \vec{p}_B - \vec{p}_A'$.

The next step is to convert $d^3\vec{p}_A'$ to spherical coordinates $d^3\vec{p}_A' = p_A'^2 \, dp_A' \, d\Omega$ and combine this w/ the energy of particle A'.

$$E_A'^2 = p_A'^2 + m_A^2$$

$$2E_A' \, dE_A' = 2p_A' \, dp_A'$$

$$E_A' \, dE_A' = \frac{d^3\vec{p}_A'}{p_A' \, d\Omega}$$

$$\Rightarrow d^3\vec{p}_A' = p_A' E_A' \, dE_A' \, d\Omega$$

Put this back into the phase space.

$$\int d(P.S._2) = \frac{1}{(4\pi)^2} \int \frac{p_A' E_A' \, dE_A' \, d\Omega}{E_A' E_B'} \delta(E_A + E_B - E_A' - E_B')$$

$$= \frac{1}{(4\pi)^2} \int \frac{p_A' \, dE_A' \, d\Omega}{E_B'} \delta(E_A + E_B - E_A' - E_B')$$

Now consider the interaction in its center of momentum frame. As such, $\vec{p}_A + \vec{p}_B = 0$ and subsequently $\vec{p}_A + \vec{p}_B - \vec{p}_A' - \vec{p}_B' = 0 \Rightarrow \vec{p}_A' = -\vec{p}_B'$. With this, we can determine the $\frac{dE_A'}{E_B'}$ ratio in the Lorentz invariant phase space as follows:

$$E_A'^2 = p_A'^2 + m_A^2 \qquad E_B'^2 = p_B'^2 + m_B^2$$

$$E_A' \, dE_A' = p_A' \, dp_A' \qquad E_B' \, dE_B' = p_B' \, dp_B'$$

$$\Rightarrow E_A' dE_A' = E_B' dE_B'$$

Define W' as the total final energy and W as the total initial energy.

$$W' \equiv E_A' + E_B' \quad ; \quad W \equiv E_A + E_B$$

$$W' = \sqrt{|\vec{p}_A'|^2 + m_A^2} + \sqrt{|\vec{p}_B'|^2 + m_B^2}$$

$$dW' = \tfrac{1}{2}(p_A'^2 + m_A^2)^{-\frac{1}{2}} 2 p_A' dp_A' + \tfrac{1}{2}(p_B'^2 + m_B^2)^{-\frac{1}{2}} 2 p_B' dp_B'$$

$$dW' = \frac{p_A' dp_A'}{E_A'} + \frac{p_B' dp_B'}{E_B'}$$

$$dW' = \left(\frac{1}{E_A'} + \frac{1}{E_B'} \right) p_A' dp_A'$$

$$dW' = \frac{W'}{E_A' E_B'} p_A' dp_A'$$

$$dW' = \frac{W'}{E_A' E_B'} E_A' dE_A'$$

$$dW' = \frac{W'}{E_B'} dE_A'$$

$$\frac{dW'}{W'} = \frac{dE_A'}{E_B'}$$

This is precisely the ratio found in the phase space. Hence,

$$\int d(P.S._2) = \frac{1}{(4\pi)^2} \int p_A' \frac{dE_A'}{E_B} d\Omega \, \delta(E_A + E_B - E_A' - E_B')$$

$$= \frac{1}{(4\pi)^2} \int p_A' \frac{dW'}{W'} d\Omega \, \delta(W - W')$$

$$\boxed{\int d(P.S._2) = \frac{1}{(4\pi)^2} \int \frac{p_A'}{W} d\Omega}$$

Finally, we can calculate the cross section using

$$p_A \cdot p_B = (E_A, \vec{p}_A') \cdot (E_B, -\vec{p}_A')$$

$$= E_A E_B + |\vec{p}_A'|^2$$

$$\Rightarrow (p_A \cdot p_B)^2 - m_A^2 m_B^2 = |\vec{p}_A'|^2 W^2$$

$$\sigma = \int d\sigma$$

$$= \frac{1}{4E_A E_B |\vec{v}|} \int |M|^2 \, d(P.S._2)$$

$$= \frac{1}{4E_A E_B |\vec{v}| (4\pi)^2} \int |M|^2 \frac{|\vec{p}_A'|}{W} \, d\Omega$$

$$= \frac{1}{4|\vec{p}_A'|W} \frac{1}{(4\pi)^2} \frac{|\vec{p}_A'|}{W} \int |M|^2 \, d\Omega$$

$$= \frac{1}{(8\pi W)^2} \int |M|^2 \, d\Omega$$

$$\Rightarrow d\sigma = \frac{1}{(8\pi W)^2} |M|^2 \, d\Omega$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi W)^2} |M|^2}$$