Field Strength Renormalization in ABC Theory

Recall that $\Pi_c^{[2]}(q^2)$ is defined by

$$-i\Pi_c^{[2]}(q^2) = (-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_A^2 + i\varepsilon} \frac{i}{(q-k)^2 - m_B^2 + i\varepsilon}$$

Now expand its value about the physical mass,

$$q^2 = m_{ph,c}^2 = m_c^2 + \Pi_c^{[2]}(m_{ph,c}^2).$$

$$\Pi_c^{[2]}(q^2) = \Pi_c^{[2]}(m_{ph,c}^2) + (q^2 - m_{ph,c}^2) \left.\frac{d\Pi_c^{[2]}}{dq^2}\right|_{q^2 = m_{ph,c}^2} + \cdots$$

Use this in the full propagator after the geometric series has been summed over.

$$\frac{i}{q^2 - m_c^2 - \Pi_c^{[2]}(m_{ph,c}^2) - (q^2 - m_{ph,c}^2) \left.\frac{d\Pi_c^{[2]}}{dq^2}\right|_{q^2 = m_{ph,c}^2} + \cdots}$$

$$= \frac{i}{(q^2 - m_{ph,c}^2)\left[1 - \left.\frac{d\Pi_c^{[2]}}{dq^2}\right|_{q^2 = m_{ph,c}^2}\right] + O(q^2 - m_{ph,c}^2)^2}$$

Consider this to be a 'corrected' propagator.

Let $|\Omega\rangle$ represent the exact vacuum. Then this corrected propagator is an approximation to the exact propagator $\langle\Omega|T[\phi_c(x_1)\phi_c(x_2)]|\Omega\rangle$.

This is opposed to the free propagator $\langle 0|T[\phi_c(x_1)\phi_c(x_2)]|0\rangle$ for a single direct line.

Consider one of the two terms in the free propagator $\oplus(t_1-t_2)\langle 0|\phi_c(x_1)\phi_c(x_2)|0\rangle$ and use the outer product to expand over all states.

$$\oplus(t_1-t_2)\sum_n \langle 0|\phi_c(x_1)|n\rangle\langle n|\phi_c(x_2)|0\rangle$$

$$= \oplus(t_1-t_2)\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega} e^{-ik\cdot(x_1-x_2)}$$

Now do the same for the exact propagator.

$$\oplus(t_1-t_2)\sum_n \langle\Omega|\phi_c(x_1)|\bar{n}\rangle\langle\bar{n}|\phi_c(x_2)|\Omega\rangle$$

where $|\bar{n}\rangle$ are the exact eigenstates of the full Hamiltonian, rather than an approximation to some order. The distinction in this case is that multiparticle states, such as the A,B loop, can appear in the $|\bar{n}\rangle$ states. Hence, such possibilities are taken into account.

This time, $\langle\Omega|\phi_c$ overlaps many states. So if $\sum_{\vec{n}}|\vec{n}\rangle\langle\vec{n}| = 1$ is still to be true, then the matrix element $\langle\Omega|\phi_c(x)|\vec{k}\rangle$ cannot have the same strength as in the free particle case, where it was the only nonzero element. Where in the free particle case

$$\langle 0|\phi_c(x)|\vec{k}\rangle = e^{-ik\cdot x}$$

we now include an arbitrary normalization constant Z_c,

$$\langle\Omega|\phi_c(x)|\vec{k}\rangle = \sqrt{Z_c}\, e^{-ik\cdot x}$$

Now we must repeat the calculation of the C-particle propagator, eqns (6.92) to (6.98).

$$\langle\Omega|T[\phi_c(x_1)\phi_c(x_2)]|\Omega\rangle$$

$$= \theta(t_1-t_2)\langle\Omega|\phi_c(x_1)\phi_c(x_2)|\Omega\rangle + \theta(t_2-t_1)\langle\Omega|\phi_c(x_2)\phi_c(x_1)|\Omega\rangle$$

$$= \theta(t_1-t_2)\sum_{\vec{n}}\langle\Omega|\phi_c(x_1)|\vec{n}\rangle\langle\vec{n}|\phi_c(x_2)|\Omega\rangle + \theta(t_2-t_1)\sum_{\vec{n}}\langle\Omega|\phi_c(x_2)|\vec{n}\rangle\langle\vec{n}|\phi_c(x_1)|\Omega\rangle$$

$$= \theta(t_1-t_2)\int Z_c e^{ik\cdot(x_1-x_2)}\, d^3\vec{k} + \theta(t_2-t_1)\int Z_c\, e^{ik\cdot(x_2-x_1)}\, d^3\vec{k}$$

$$= Z_c\int\frac{d^3\vec{k}}{(2\pi)^3 2w}\left[\theta(t_1-t_2)e^{ik\cdot(x_1-x_2)} + \theta(t_2-t_1)e^{ik\cdot(x_2-x_1)}\right]$$

The rest here is an exact recitation of pages 4-5 of the C-propagator tab.

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x_1-x_2)} \left[\frac{i Z_c}{k^2 - m_{ph,c}^2 + i\varepsilon} + \text{loop contributions} \right]$$

Directly compare the Fourier Transform of this to what we saw on page 2:

$$\frac{i}{(q^2 - m_{ph,c}^2)\left[1 - \left.\frac{d\Pi_c^{[2]}}{dq^2}\right|_{q^2=m_{ph,c}^2}\right] + O(q^2 - m_{ph,c}^2)^2}$$

and to order q^2 we can make the approximation

$$\left[1 - \frac{d\Pi_c^{[2]}}{dq^2}\right]^{-1} \cong 1 + \frac{d\Pi_c^{[2]}}{dq^2}$$

Upon comparison, we see that to this order the renormalization constant is

$$\boxed{Z_c \approx Z_c^{[2]} = 1 + \left.\frac{d\Pi_c^{[2]}}{dq^2}\right|_{q^2=m_{ph,c}^2}}$$