Contour Integration

Consider a particle moving in the x,y-plane subject to a force

$$\vec{F} = (P(x,y), Q(x,y))$$

where both components of the force vary, in general, throughout the plane.

Suppose now that the particle around a closed path C while subject to $\vec{F}$. The total work performed by the force is

$$W_c = \oint_C \vec{F} \cdot d\vec{r}$$

$$W_c = \oint_C [P\,dx + Q\,dy]$$

Stokes Theorem ($\int_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$) is particularly applicable here.

$$W_c = \iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$$

$$= \iint_S \left(\frac{dQ}{dx} - \frac{dP}{dy} \right) dx\,dy$$

If the force is conservative, it is derivable as the gradient of some scalar potential function $V(x,y)$. Therefore

$$\begin{cases} F_x = P(x,y) = -\dfrac{\partial V}{\partial x} \\ F_y = Q(x,y) = -\dfrac{\partial V}{\partial y} \end{cases}$$

are conditions for a conservative force. Furthermore, since the order of derivatives does not matter, it is also true that

$$\cancel{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Hence, without even defining $\vec{F}$ we know that if we assume it to be conservative

$$W_c = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy$$

$$\boxed{W_c = 0}$$

There is, however, room for exception. Consider the particular potential

$$V(x,y) = -\arctan\frac{y}{x}$$

$$\Rightarrow \begin{cases} P = -\dfrac{\partial V}{\partial x} = \dfrac{-y}{x^2+y^2} \\ Q = -\dfrac{\partial V}{\partial y} = \dfrac{x}{x^2+y^2} \end{cases}$$

$$\Rightarrow \frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}$$

We see that this potential does specify a conservative force. However, if we assume

$$W = \oint_C$$

the path C to be the unit circle

$$W_c = \oint_c [P\,dx + Q\,dy]$$

$$= \oint_c [-\sin\theta(-\sin\theta\,d\theta) + \cos\theta(\cos\theta)\,d\theta]$$

$$= \oint_c [\sin^2\theta + \cos^2\theta]\,d\theta$$

$$= \oint_c d\theta$$

$$= 2\pi$$

So what's the problem? This is clearly a nonzero value even though, in terms of angles, an addition of 2π is not a distinct value.

~~This force~~

The potential, although conservative, is ill-defined b/c it's gradient, (the force), is not single-valued.

Such a force is unphysical b/c it is self-contradictory. Therefore, the potential requires the additional property of being analytic.

A function is <u>analytic</u> in some region if it has a unique derivative at every point in that region.

Let's now move into the complex plane. Let $z = x + iy$ and the function $f(z)$ be analytic in a region R of the complex plane. The derivative of $f(z)$ at a point z in the complex plane is defined by

$$\frac{df}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

Note that here, as opposed to a derivative in $\mathbb{R}$, Δz contains two parameters x and y. Therefore, Δz is actually a vector and there are infinitely many possible directions to approach the point z. Since $f(z)$ is analytic, it does not matter which direction z is approached from.

Now let $u(x,y)$ and $v(x,y)$ be the real and imaginary parts of f. As ~~it does not matter which direction~~ the direction from which z is approached does not matter, let's take the derivative from purely the real direction ($\Delta z = \Delta x$) and from purely the imaginary direction ($\Delta z = \Delta y$) and compare the two results.

If $\Delta z = \Delta x$, $\Delta y = 0$:

$$\frac{df}{dz} = \lim_{\Delta x \to 0} \left[\frac{u(x+\Delta x, y) - u(x,y) + iv(x+\Delta x, y) - iv(x,y)}{\Delta x} \right]$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

If $\Delta y = \Delta y$, $\Delta x = 0$:

$$\frac{df}{dx} = \lim_{\Delta y \to 0} \left[\frac{u(x, y+\Delta y) - u(x,y) + i v(x, y+\Delta y) - i v(x,y)}{\Delta y} \right]$$

$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

We already know by analycity that the two results must be equal.

$$\Rightarrow \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

and matching real and imaginary parts we have the Cauchy - Riemann Relations

$$\boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \end{aligned}}$$

Next, analogous to the real case, consider a closed path integral in the complex plane,

$$I = \oint_C f(y)\, dy$$

$$I = \oint (u\,dx - v\,dy) + i \oint (v\,dx + u\,dy)$$

For the sake of argument, what would be required for the integral to vanish?

~~$u\,dx - v\,dy = 0$~~

~~$u\,dx = v\,dy$~~

Note that each of these two integrals are now real, so the arguments from page 1, 2 apply. We then find that it is just the ~~Cauchy~~ Cauchy-Riemann Relations themselves that allow the integral to vanish, but these were already assumed true in the first place. Hence, quite generally

$$\boxed{\oint_C f(y)\,dy = 0}$$ Cauchy's Theorem

if $f(y)$ is analytic.

~~Consider~~

Example: Let's consider $f(z) = \frac{1}{z}$, which is NOT analytic at $z=0$. Integrate this around a circle of radius ρ centered on the origin. On this path $z = \rho e^{i\theta}$.

$$\oint_C \frac{dz}{z} = \oint_C \frac{\rho i e^{i\theta} \, d\theta}{\rho e^{i\theta}}$$

$$= \oint_C i \, d\theta$$

$$= 2\pi i$$

Let's take a step back. Recall how in the derivation of Cauchy's Theorem we used Stokes Theorem in the two real integrals to go from

$$I = \oint (u\,dx - v\,dy) + i \oint (v\,dx + u\,dy)$$

to

$$\oint f(z)\,dz = 0$$

f was assumed analytic, so when we used Stokes Theorem and changed the single path integral into a ~~double~~ surface integral, the analycity of the integrand over that surface is implied. Thus, it is b/c of Stokes Theorem that Cauchy's Theorem holds for any closed path.

Theorem: Cauchy's Integral Formula

Let $f(z)$ be analytic inside and on a simple closed curve C that encloses some point $z=a$. Then

$$\oint \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

Proof: $f(z)$ may be analytic at $z=a$, but if we let the integrand to instead have the form $\frac{f(z)}{z-a}$, then the integrand is NOT analytic specifically and solely at $z=a$.

Since the shape of C does not affect the outcome, we may assume it to be equivalent to a circle centered (always) on $z=a$, and for the same reason we may shrink the size of the circle. On the circle, z is given by $z = a + \rho e^{i\theta}$.

$$\oint_C \frac{f(z)}{z-a}\,dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})\, i\rho e^{i\theta}\, d\theta}{\rho e^{i\theta}}$$

$$= i\int_0^{2\pi} f(a+\rho e^{i\theta})\, d\theta$$

$$= i\int_0^{2\pi} f(a)\, d\theta \quad \text{as} \quad \rho \to 0$$

$$= \tfrac{1}{2}\, 2\pi i\, f(a) \quad \text{as} \quad \rho \to 0.$$