Use $H = \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p$ and $[a_p, a_{p'}^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}')$

to calculate the following commutation relationship:

$$[H, a_p^\dagger] = H a_p^\dagger - a_p^\dagger H$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p a_p^\dagger - a_p^\dagger \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger \left[(2\pi)^3 \delta^3(\vec{p} - \vec{p}') + a_{p'}^\dagger a_p \right] - a_p^\dagger \int \frac{d^3p}{(2\pi)^3} \omega_p \left[a_p a_{p'}^\dagger - (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \right]$$

$$= \omega_{p'} a_{p'}^\dagger + \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_{p'}^\dagger a_p - \int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p a_{p'}^\dagger + a_{p'}^\dagger \omega_{p'}$$

$$= 2\omega_{p'} a_{p'}^\dagger + \int \frac{d^3p}{(2\pi)^3} \omega_p \left[a_p^\dagger a_{p'}^\dagger a_p - a_p^\dagger a_p a_{p'}^\dagger \right]$$

$$= 2\omega_{p'} a_{p'}^\dagger + \int \frac{d^3p}{(2\pi)^3} \omega_p \left[a_p^\dagger \left(a_p a_{p'}^\dagger - (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \right) - a_p^\dagger a_p a_{p'}^\dagger \right]$$

$$= 2\omega_{p'} a_{p'}^\dagger + \cancel{\int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p a_{p'}^\dagger} - \cancel{\int \frac{d^3p}{(2\pi)^3} \omega_p a_p^\dagger a_p a_{p'}^\dagger} - \omega_{p'} a_{p'}^\dagger$$

$$\boxed{[H, a_p^\dagger] = \omega_{p'} a_{p'}^\dagger}$$

2a) Use $P^\mu = \int \frac{d^3p}{(2\pi)^3} p^\mu a_p^\dagger a_p$ and $[a_p, a_{p'}^\dagger] = (2\pi)^3 \delta^3(\vec{p} - \vec{p}')$ to calculate the following commutation relationship:

$$[P^\mu, a_k] = P^\mu a_k - a_k P^\mu$$

$$= \int \frac{d^3p}{(2\pi)^3} p^\mu a_p^\dagger a_p a_k - a_k \int \frac{d^3p}{(2\pi)^3} p^\mu a_p^\dagger a_p$$

$$= \int \frac{d^3p}{(2\pi)^3} p^\mu a_p^\dagger a_p a_k - \int \frac{d^3p}{(2\pi)^3} p^\mu a_k a_p^\dagger a_p$$

$$= \int \frac{d^3p}{(2\pi)^3} p^\mu a_p^\dagger a_p a_k - \int \frac{d^3p}{(2\pi)^3} p^\mu \left[(2\pi)^3 \delta^3(k-p) + a_p^\dagger a_k\right] a_p$$

$$= \int \frac{d^3p}{(2\pi)^3} p^\mu a_p^\dagger a_p a_k - k^\mu a_k - \int \frac{d^3p}{(2\pi)^3} p^\mu a_p^\dagger a_k a_p$$

Now use $[a_k, a_p] = 0$.

$$= \cancel{\int \frac{d^3p}{(2\pi)^3} p^\mu a_p^\dagger a_p a_k} - k^\mu a_k - \cancel{\int \frac{d^3p}{(2\pi)^3} p^\mu a_p^\dagger a_p a_k}$$

$$\boxed{[P^\mu, a_k] = -k^\mu a_k}$$

Knowing the previous result, note the following:

$$[P^\mu, a_k] = -k^\mu a_k$$

$$P^\mu a_k - a_k P^\mu = -k^\mu a_k$$

$$P^\mu a_k = a_k P^\mu - k^\mu a_k$$

$$P^\mu a_k = a_k (P^\mu - k^\mu)$$

The commutation relationship has taken the form of an eigenvalue problem with P^μ as the operator and $(P^\mu - k^\mu)$ its eigenvalue.

Next, recall that there is a theorem from nonrelativistic quantum mechanics that says if

$$X|x\rangle = x|x\rangle$$

then we can also say for a function $f(x)$

$$f(X)|x\rangle = f(x)|x\rangle.$$

Using this property:

$$e^{iP\cdot x} a_k e^{-iP\cdot x} = a_k e^{i(P^\mu - k^\mu)\cdot x_\mu} e^{-iP^\mu \cdot x_\mu}$$

$$= a_k e^{i(P^\mu - k^\mu - P^\mu)\cdot x_\mu}$$

$$\boxed{e^{iP\cdot x} a_k e^{-iP\cdot x} = a_k e^{-ik\cdot x}}$$

Use the quantized normal mode expansion

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$[P^\nu, \phi(x)] = \int \frac{d^3p'}{(2\pi)^3} p^\nu a_{p'}^\dagger a_{p'} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$- \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right] \int \frac{d^3p'}{(2\pi)^3} p^\nu a_{p'}^\dagger a_{p'}$$

$$[P^\nu, \phi(x)] = \iint \frac{d^3p'\, d^3p\, p^\nu}{(2\pi)^6 \sqrt{2\omega_p}} \left[a_{p'}^\dagger a_{p'} a_p e^{i\vec{p}\cdot\vec{x}} + a_{p'}^\dagger a_{p'} a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$- \iint \frac{d^3p'\, d^3p\, p^\mu}{(2\pi)^6 \sqrt{2\omega_p}} \left[a_p a_{p'}^\dagger a_{p'} e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger a_{p'}^\dagger a_{p'} e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$[P^\nu, \phi(x)] = \iint \frac{d^3p'\, d^3p\, p^\nu}{(2\pi)^6 \sqrt{2\omega_p}} \left[a_{p'}^\dagger a_{p'} a_p e^{i\vec{p}\cdot\vec{x}} + a_{p'}^\dagger a_{p'} a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$- \iint \frac{d^3p'\, d^3p\, p^\nu}{(2\pi)^6 \sqrt{2\omega_p}} \left[\left((2\pi)^3 \delta^3(\vec{p}-\vec{p}') + a_{p'}^\dagger a_p \right) a_{p'} e^{i\vec{p}\cdot\vec{x}} + a_{p'}^\dagger a_p^\dagger a_{p'} e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$[P^\nu, \phi(x)] = \iint \frac{d^3p'\, d^3p\, p^\nu}{(2\pi)^6 \sqrt{2\omega_p}} \left[a_{p'}^\dagger a_{p'} a_p e^{i\vec{p}\cdot\vec{x}} + a_{p'}^\dagger a_{p'} a_p^\dagger e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$- \iint \frac{d^3p'\, d^3p\, p^\nu}{(2\pi)^6 \sqrt{2\omega_p}} \left[(2\pi)^3 \delta^3(\vec{p}-\vec{p}') a_{p'} e^{i\vec{p}\cdot\vec{x}} + a_{p'}^\dagger a_p a_{p'} e^{i\vec{p}\cdot\vec{x}} + a_{p'}^\dagger a_p^\dagger a_{p'} e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$[P^\mu, \phi(x)] = \iint \frac{d^3p'\, d^3p\, p^\mu}{(2\pi)^6 \sqrt{2\omega_p}} \left[a^\dagger_{p'} a_{p'} a_p e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{p'} a_{p'} a^\dagger_p e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$+ \int \frac{d^3p'}{(2\pi)^3} \frac{p^\mu}{\sqrt{2\omega_{p'}}} a_{p'} e^{i\vec{p}'\cdot\vec{x}} - \iint \frac{d^3p'\, d^3p\, p^\mu}{(2\pi)^6 \sqrt{2\omega_p}} \left[a^\dagger_{p'} a_p a_{p'} e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{p'} a^\dagger_p a_{p'} e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$[P^\mu, \phi(x)] = \iint \frac{d^3p'\, d^3p\, p^\mu}{(2\pi)^6 \sqrt{2\omega_p}} \left[a^\dagger_{p'} a_{p'} a_p e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{p'} a_{p'} a^\dagger_p e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$+ \int \frac{d^3p'}{(2\pi)^3} \frac{p^\mu}{\sqrt{2\omega_{p'}}} a_{p'} e^{i\vec{p}'\cdot\vec{x}} - \iint \frac{d^3p'\, d^3p\, p^\mu}{(2\pi)^6 \sqrt{2\omega_p}} \left[a^\dagger_{p'} a_{p'} a_p e^{i\vec{p}\cdot\vec{x}} \right.$$

$$\left. + a^\dagger_{p'} \left(a_{p'} a^\dagger_p - (2\pi)^3 \delta^3(\vec{p} - \vec{p}') \right) e^{-i\vec{p}\cdot\vec{x}} \right]$$

$$[P^\mu, \phi(x)] = \cancel{\iint \frac{d^3p'\, d^3p\, p^\mu}{(2\pi)^6 \sqrt{2\omega_p}} \left[a^\dagger_{p'} a_{p'} a_p e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{p'} a_{p'} a^\dagger_p e^{-i\vec{p}\cdot\vec{x}} \right]}$$

$$+ \int \frac{d^3p'\, p^\mu}{(2\pi)^3 \sqrt{2\omega_{p'}}} a_{p'} e^{i\vec{p}'\cdot\vec{x}} - \cancel{\iint \frac{d^3p'\, d^3p\, p^\mu}{(2\pi)^6 \sqrt{2\omega_p}} \left[a^\dagger_{p'} a_{p'} a_p e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{p'} a_{p'} a^\dagger_p e^{-i\vec{p}\cdot\vec{x}} \right]}$$

$$+ \int \frac{d^3p'\, p^\mu}{(2\pi)^3 \sqrt{2\omega_{p'}}} a^\dagger_{p'} e^{-i\vec{p}'\cdot\vec{x}}$$

$$[P^\mu, \phi(x)] = \int \frac{d^3p'\, p^\mu}{(2\pi)^3 \sqrt{2\omega_{p'}}} \left[a_{p'} e^{i\vec{p}'\cdot\vec{x}} + a^\dagger_{p'} e^{-i\vec{p}'\cdot\vec{x}} \right]$$

$$[P^\mu, \phi(x)] = p^\mu \phi(x)$$

$$\boxed{[P^\mu, \phi(x)] = -i\partial^\mu \phi(x)}$$

2b) There is a very slick way to do this problem. First, take note of equations (2.41) and (2.42) from Peskin-Schroeder:

$$\phi(\vec{x})|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-i\vec{p}\cdot\vec{x}} |\vec{p}\rangle$$

$$\Rightarrow \underline{\langle 0|\phi(x)|\vec{p}\rangle = e^{i\vec{p}\cdot\vec{x}}}$$

Now exploit this simple relationship.

$$\langle K|\phi(x)\phi(y)|K\rangle = \langle K|\phi(x)|0\rangle\langle 0|\phi(y)|K\rangle$$

$$= e^{-iK\cdot x} e^{iK\cdot y}$$

$$= e^{-iK\cdot(x-y)}$$

$$= e^{-iK\cdot(x-y)} e^{iK\cdot 0}$$

$$= \langle K|\phi(x-y)|0\rangle\langle 0|\phi(0)|K\rangle$$

$$\boxed{\langle K|\phi(x)\phi(y)|K\rangle = \langle K|\phi(x-y)\phi(0)|K\rangle}$$