7.1 The Complex Scalar Field: Global U(1) phase Invariance, Particle and Anti-Particles

Postulate a Lagrangian for two free fields ϕ_1 and ϕ_2 having the same mass M.

$$\mathcal{L} = \left[\frac{1}{2}\partial_\mu\phi_1\,\partial^\mu\phi_1 - \frac{1}{2}M^2\phi_1^2\right] + \left[\frac{1}{2}\partial_\mu\phi_2\,\partial^\mu\phi_2 - \frac{1}{2}M^2\phi_2^2\right]$$

'Particle' and 'Antiparticle' are distinguished by having opposite values of a conserved quantum number (or numbers). Remember that quantum numbers are eigenvalues, with corresponding eigenoperators that determine them. Furthermore, since these quantum numbers are conserved their operators commute w/ the Hamiltonian and are constant in time. These operators are called <u>symmetry operators</u>.

The above Lagrangian density is symmetric under an O(2) transformation; or in other words, a 2-dim rotation.

This means that $\mathcal{L}$ is invariant if ϕ_1 and ϕ_2 are replaced by ϕ_1' and ϕ_2'

$$\begin{cases} \phi_1' = (\cos\alpha)\phi_1 - (\sin\alpha)\phi_2 \\ \phi_2' = (\sin\alpha)\phi_1 + (\cos\alpha)\phi_2 \end{cases}$$

Consider rotation through an infinitesimal angle ε

$$\begin{cases} \phi_1' = (\cos\varepsilon)\phi_1 - (\sin\varepsilon)\phi_2 \\ \phi_2' = (\sin\varepsilon)\phi_1 + (\cos\varepsilon)\phi_2 \end{cases}$$

$$\Rightarrow \begin{cases} \phi_1' = \phi_1 - \varepsilon\phi_2 \\ \phi_2' = \phi_2 + \varepsilon\phi_1 \end{cases}$$

Thus, we can define the changes in field value by $\delta\phi_1 \equiv -\varepsilon\phi_2$ and $\delta\phi_2 = \varepsilon\phi_1$, or

$$\begin{cases} \delta\phi_1 = \phi_1' - \phi_1 \\ \delta\phi_2 = \phi_2' - \phi_2 \end{cases}$$

Next, as $\mathcal{L}$ is invariant under the $O(2)$ transformation, it is equivalent to say $\delta\mathcal{L} = 0$, where $\delta\mathcal{L}$ is the change in the value of the Lagrangian density

due to an infinitesimal change to its parameters. In this case, $\delta\mathcal{L}$ corresponds to ϕ_1 changing by $\delta\phi_1$ and ϕ_2 changing by $\delta\phi_2$.

Note that $\mathcal{L}$ is an explicit function of ϕ_1, ϕ_2, $\partial_\mu\phi_1$, and $\partial_\mu\phi_2$. As such, we may write the general expression:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\delta(\partial_\mu\phi_1) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\delta(\partial_\mu\phi_2) + \frac{\partial\mathcal{L}}{\partial\phi_1}\delta(\phi_1) + \frac{\partial\mathcal{L}}{\partial\phi_2}\delta\phi_2 = 0$$

Use the Euler-Lagrange equation for these fields to make substitutions.

$$0 = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\delta(\partial_\mu\phi_1) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\delta(\partial_\mu\phi_2)$$
$$+ \left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\right)\right]\delta\phi_1 + \left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\right)\right]\delta\phi_2$$

And since $\delta(\partial_\mu\phi_i) = \partial_\mu(\delta\phi_i)$

$$0 = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\partial_\mu(\delta\phi_1) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\partial_\mu(\delta\phi_2)$$
$$+ \left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\right)\right]\delta\phi_1 + \left[\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\right)\right]\delta\phi_2$$

By Product Rule:

$$0 = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \delta\phi_1 \right] + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \delta\phi_2 \right]$$

$$0 = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \delta\phi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \delta\phi_2 \right]$$

Note that the steps taken here are completely general; the actual form of $\mathcal{L}$ has not been taken into account. Let's do that now and substitute $\delta\phi_1 = -\varepsilon\phi_2$ and $\delta\phi_2 = \varepsilon\phi_1$:

$$0 = \partial_\mu \left[-\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \varepsilon\phi_2 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \varepsilon\phi_1 \right]$$

Directly calculate the derivatives $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}$ from the given $\mathcal{L}$.

$$0 = \varepsilon\, \partial_\mu \left[(\partial^\mu \phi_2)\phi_1 - (\partial^\mu \phi_1)\phi_2 \right]$$

From here, define $N_\phi^\mu \equiv (\partial^\mu \phi_2)\phi_1 - (\partial^\mu \phi_1)\phi_2$.

$$0 = \varepsilon\, \partial_\mu N_\phi^\mu$$

But ε is arbitrary, so we have found the symmetry operator corresponding to the postulated $\mathcal{L}$.

$$\boxed{\partial_\mu N_\phi^\mu = 0}$$

To see how N_ϕ distinguishes 'particle' from 'antiparticle', consider ϕ_1 and ϕ_2 to be components of a single complex field.

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

$$\phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

Recall the normal mode expansion of ϕ_1, ϕ_2:

$$\phi_i(x) = \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega}} \left[a_i(\vec{k}) e^{-ik\cdot x} + a_i^\dagger(\vec{k}) e^{ik\cdot x} \right] \qquad (i = 1, 2)$$

Use these to compute the expansion of ϕ.

From here, define $N_\phi^\mu \equiv (\partial^\mu \phi_2)\phi_1 - (\partial^\mu \phi_1)\phi_2$.

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Use these to compute the expansion of ϕ.

$$\phi = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega}} \left[a_1(k) e^{-ik\cdot x} + a_1^\dagger(k) e^{ik\cdot x} \right]$$

$$- \frac{i}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega}} \left[a_2(k) e^{-ik\cdot x} + a_2^\dagger(k) e^{ik\cdot x} \right]$$

$$\phi = \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega}} \left[(a_1 - i a_2) e^{-ik\cdot x} + (a_1^\dagger - i a_2^\dagger) e^{ik\cdot x} \right]$$

Define $a \equiv \frac{1}{\sqrt{2}}(a_1 - i a_2)$; $b^\dagger \equiv \frac{1}{\sqrt{2}}(a_1 - i a_2)$.

$$\phi = \int \frac{d^3\vec{k}}{(2\pi)^3 \sqrt{2\omega}} \left[a(k) e^{-ik\cdot x} + b^\dagger(k) e^{ik\cdot x} \right]$$

Remember the O(2) transformation?

Now it is a rotation in the complex plane, which is also a phase change

$$\phi' = e^{-i\alpha} \phi$$

Thus, we have arrived at the notion of "phase invariance." Making the proper substitutions, the symmetry current is

$$N_\phi^\mu = i(\phi^\dagger \partial^\mu \phi - \phi \partial^\mu \phi^\dagger)$$

$$N_\phi^\mu = \int \frac{d^3\vec{k}}{(2\pi)^3} \left[a^\dagger(k) a(k) - b^\dagger(k) b(k) \right]$$

N_ϕ is the difference b/w the number operators for 'a' quanta and for 'b' quanta. So a 'b' particle is the antiparticle of an 'a' particle.

The Complex Field Propagator

The calculation of this propagator is nearly identical to the case for the ABC Theory propagator.

$$\langle 0|T[\phi(x_1)\phi^\dagger(x_2)]|0\rangle = \begin{cases} \langle 0|\phi(x_1)\phi^\dagger(x_2)|0\rangle & \text{for } t_1 > t_2 \\ \langle 0|\phi^\dagger(x_2)\phi(x_1)|0\rangle & \text{for } t_2 > t_1 \end{cases}$$

$$= \langle 0|\left[\Theta(t_1-t_2)\phi(x_1)\phi^\dagger(x_2) + \Theta(t_2-t_1)\phi^\dagger(x_2)\phi(x_1)\right]|0\rangle$$

$$= \langle 0|\Theta(t_1-t_2)\int\frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega}}\left[a(\vec{k})e^{-i\vec{k}\cdot x_1} + b^\dagger(\vec{k})e^{i\vec{k}\cdot x_1}\right]\left[\int\frac{d^3\vec{k}'}{(2\pi)^3\sqrt{2\omega'}}\left[a^\dagger(\vec{k}')e^{i\vec{k}'\cdot x_2} + b(\vec{k}')e^{-i\vec{k}'\cdot x_2}\right]\right]|0\rangle$$

$$+ \langle 0|\Theta(t_2-t_1)\int\frac{d^3\vec{k}'}{(2\pi)^3\sqrt{2\omega'}}\left[a^\dagger(\vec{k}')e^{i\vec{k}'\cdot x_2} + b(\vec{k}')e^{-i\vec{k}'\cdot x_2}\right]\left[\int\frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega}}\left[a(\vec{k})e^{-i\vec{k}\cdot x_1} + b^\dagger(\vec{k})e^{i\vec{k}\cdot x_1}\right]\right]|0\rangle$$

$$= \langle 0|\Theta(t_1-t_2)\iint\frac{d^3\vec{k}\,d^3\vec{k}'}{(2\pi)^6 2\sqrt{\omega\omega'}}\Big[a(\vec{k})a^\dagger(\vec{k}')e^{-i\vec{k}\cdot x_1}e^{i\vec{k}'\cdot x_2} + \cancelto{0}{a(\vec{k})b(\vec{k}')}e^{-i\vec{k}\cdot x_1}e^{-i\vec{k}'\cdot x_2}$$

$$+ \cancelto{0}{b^\dagger(\vec{k})a^\dagger(\vec{k}')}e^{i\vec{k}\cdot x_1}e^{i\vec{k}'\cdot x_2} + \cancelto{0}{b^\dagger(\vec{k})b(\vec{k}')}e^{i\vec{k}\cdot x_1}e^{-i\vec{k}'\cdot x_2}\Big]|0\rangle$$

$$+ \langle 0|\Theta(t_2-t_1)\iint\frac{d^3\vec{k}'\,d^3\vec{k}}{(2\pi)^3 2\sqrt{\omega\omega'}}\Big[\cancelto{0}{a^\dagger(\vec{k}')a(\vec{k})}e^{i\vec{k}'\cdot x_2}e^{-i\vec{k}\cdot x_1} + \cancelto{0}{a^\dagger(\vec{k}')b^\dagger(\vec{k})}e^{i\vec{k}'\cdot x_2}e^{i\vec{k}\cdot x_1}$$

$$+ \cancelto{0}{b(\vec{k}')a(\vec{k})}e^{-i\vec{k}'\cdot x_2}e^{i\vec{k}\cdot x_1} + b(\vec{k}')b^\dagger(\vec{k})e^{-i\vec{k}'\cdot x_2}e^{i\vec{k}\cdot x_1}\Big]|0\rangle$$

$$= \langle 0|\Theta(t_1-t_2)\iint\frac{d^3\vec{k}\,d^3\vec{k}'}{(2\pi)^6 2\sqrt{\omega\omega'}}\left[a(\vec{k})a^\dagger(\vec{k}')e^{-i\vec{k}\cdot x_1}e^{i\vec{k}'\cdot x_2}\right]|0\rangle$$

$$+ \langle 0|\Theta(t_2-t_1)\iint\frac{d^3\vec{k}'\,d^3\vec{k}}{(2\pi)^6 2\sqrt{\omega\omega'}}\left[b(\vec{k}')b^\dagger(\vec{k})e^{-i\vec{k}'\cdot x_2}e^{i\vec{k}\cdot x_1}\right]|0\rangle$$

Now use the commutation relations

$$[a(\vec{k}), a^\dagger(\vec{k}')]_- = (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$\Rightarrow a(\vec{k})a^\dagger(\vec{k}') = (2\pi)^3\delta^3(\vec{k}-\vec{k}') + a^\dagger(\vec{k}')a(\vec{k})$$

$$[b(\vec{k}), b^\dagger(\vec{k}')]_- = (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

$$\Rightarrow b(\vec{k})b^\dagger(\vec{k}') = (2\pi)^3\delta^3(\vec{k}-\vec{k}') + b^\dagger(\vec{k}')b(\vec{k})$$

$$= \langle 0|\Theta(t_1-t_2)\iint \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^6 2\sqrt{\omega\omega'}}\left[(2\pi)^3\delta^3(\vec{k}-\vec{k}')e^{-i\vec{k}\cdot x_1}e^{i\vec{k}'\cdot x_2}\right]_-|0\rangle$$

$$+ \langle 0|\Theta(t_2-t_1)\iint \frac{d^3\vec{k}'\, d^3\vec{k}}{(2\pi)^6 2\sqrt{\omega\omega'}}\left[(2\pi)^3\delta^3(\vec{k}-\vec{k}')e^{-i\vec{k}'\cdot x_2}e^{i\vec{k}\cdot x_1}\right]_-|0\rangle$$

$$= \langle 0|\Theta(t_1-t_2)\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega} e^{-i\vec{k}\cdot x_1}e^{i\vec{k}\cdot x_2}|0\rangle \langle 0|\Theta(t_2-t_1)\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega} e^{-i\vec{k}\cdot x_2}e^{i\vec{k}\cdot x_1}|0\rangle$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega}\left[\Theta(t_1-t_2)e^{-i\vec{k}\cdot(x_1-x_2)} + \Theta(t_2-t_1)e^{i\vec{k}\cdot(x_1-x_2)}\right]_-$$

Getting from here to the propagator is identical to the ABC-Theory method, so refer to that tab:

$$\boxed{\langle 0|T[\phi(x_1)\phi^\dagger(x_2)]|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x_1-x_2)} \frac{i}{k^2-M^2+i\varepsilon}}$$

Or, in momentum-space

$$\boxed{\frac{i}{k^2-M^2+i\varepsilon}}$$