The Klein-Gordon Propagator

We now know

$$[\phi(x),\phi(y)] = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right]$$

So, since the right-hand-side has no operators, we may also say

$$\langle 0|[\phi(x),\phi(y)]|0\rangle = [\phi(x),\phi(y)]$$

$$= \int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2E_p}e^{-ip\cdot(x-y)}\Big|_{p^0=E_p} + \frac{1}{-2E_p}e^{-ip\cdot(x-y)}\Big|_{p^0=-E_p}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2E_p}\oint \frac{dp^0}{2\pi i}\frac{e^{-ip\cdot(x-y)}}{p-E_p} - \frac{1}{2E_p}\oint\frac{dp^0}{2\pi i}\frac{e^{-ip\cdot(x-y)}}{p-(-E_p)}\right]$$

where I used Cauchy's Integral Formula

$$\oint \frac{f(z)dz}{z-a} = 2\pi i f(a)$$

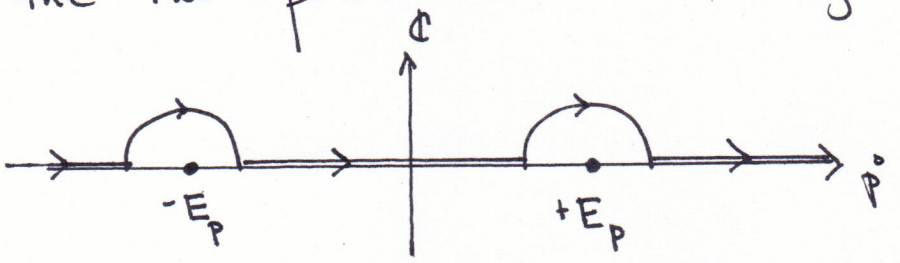
and $f(z) = e^{ip\cdot(x-y)}$, $z = p$, $a = \pm E_p$. So we have poles at $p^0 = \pm E_p$.

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\oint\frac{dp^0}{2\pi i}e^{-ip\cdot(x-y)}\left[\frac{1}{p^0-E_p} - \frac{1}{p^0+E_p}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\oint\frac{dp^0}{2\pi i}e^{-ip\cdot(x-y)}\left[\frac{-2E_p}{p^2-m^2}\right]$$

$$= \int \frac{d^3p}{(2\pi)^3} \oint \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

Note the two poles we must integrate around



If $x^0 > y^0$, the contour closes below picking up the two poles and yielding a finite answer. If the contour closes above, it contains no poles and is zero.

To take both the $x^0 > y^0$ and $x^0 < y^0$ cases into account, we can use the Heaviside Step Function to define

$$D_R(x-y) \equiv \Theta(x^0 - y^0) \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$$

$$= \Theta(x^0 - y^0) \int \frac{d^3p}{(2\pi)^3} \oint \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

The quantity $D_R(x-y)$ is a Green's Function of the Klein-Gordon operator. This means that when $(\partial_\mu \partial^\mu + m^2)$ is applied to $D_R(x-y)$ we get a Dirac δ-function.

$$(\partial^2+m^2)D_R(x-y) = \partial^2\left[\theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle\right] + m^2\theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$= \left(\partial^2\theta(x^0-y^0)\right)\langle 0|[\phi(x),\phi(y)]|0\rangle + 2\left(\partial_\mu\theta(x^0-y^0)\right)\partial^\mu\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ \theta(x^0-y^0)(\partial^2+m^2)\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$= -\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle + 2\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle + 0$$

$$= i\delta^{(4)}(x-y)$$

Thus, with $(\partial^2+m^2)D_R(x-y) = i\delta^{(4)}(x-y)$ a Fourier Transform leads to the same result.

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \tilde{D}_R(p)$$

$$(\partial^2+m^2)D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} (\partial^2+m^2) e^{-ip\cdot(x-y)} \tilde{D}_R(p)$$

$$i\delta^{(4)}(x-y) = \int \frac{d^4p}{(2\pi)^4} (\partial^2+m^2) e^{-ip\cdot(x-y)} \tilde{D}_R(p)$$

Compare this to the Green's Function above to find $\tilde{D}_R(p)$.

$$i\delta^{(4)}(x-y) = (\partial^2+m^2)D_R(x-y)$$

$$= (\partial^2 + m^2) \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i(\partial^2 + m^2)}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

Hence,

$$\tilde{D}_R(p) = \frac{i}{p^2 - m^2}$$

And putting this back into the Fourier expansion of $D_R(x-y)$, we get the same result.

$$\boxed{D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} e^{-ip\cdot(x-y)}}$$