2.1) If classical electromagnetism (for no sources) follows from the action

$$S = \int d^4x \left(\frac{-1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

for $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

then the Lagrangian density is the integrand.

$$\Rightarrow \mathcal{L} = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu}$$

We may then employ the Euler-Lagrange equation to derive Maxwell's Equations of motion. Specifically, since the Lagrangian density contains terms with both raised and lowered indices, we want the two Euler-Lagrange equations

$$\text{(I)} \qquad \frac{\partial \mathcal{L}}{\partial A_\mu} - \partial_\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} \right] = 0$$

$$\text{(II)} \qquad \frac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} \right] = 0$$

Consider (I) for now. First, I want to rewrite the Lagrangian density in a more convenient form.

$$\mathcal{L} = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\mathcal{L} = \frac{-1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\mathcal{L} = \frac{-1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(g^{\alpha\mu} g^{\beta\nu} \partial_\alpha A_\beta - g^{\alpha\mu} g^{\beta\nu} \partial_\beta A_\alpha\right)$$

$$\mathcal{L} = \frac{-1}{4}\left[(\partial_\mu A_\nu)(g^{\alpha\mu} g^{\beta\nu} \partial_\alpha A_\beta) - (\partial_\mu A_\nu)(g^{\alpha\mu} g^{\beta\nu} \partial_\beta A_\alpha) - (\partial_\nu A_\mu)(g^{\alpha\mu} g^{\beta\nu} \partial_\alpha A_\beta) + (\partial_\nu A_\mu)(g^{\alpha\mu} g^{\beta\nu} \partial_\beta A_\alpha)\right]$$

With this, I'm ready to start calculating the pieces of the Euler-Lagrange equation. The first is trivial, and indicative of there being no sources.

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0$$

This other term is not so trivial.

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \left(\frac{-1}{4}\right)\frac{\partial}{\partial(\partial_\nu A_\mu)}\Big[(\partial_{\bar\mu} A_{\bar\nu})(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \partial_{\bar\alpha} A_{\bar\beta}) - (\partial_{\bar\mu} A_{\bar\nu})(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \partial_{\bar\beta} A_{\bar\alpha})$$
$$- (\partial_{\bar\nu} A_{\bar\mu})(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \partial_{\bar\alpha} A_{\bar\beta}) + (\partial_{\bar\nu} A_{\bar\mu})(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \partial_{\bar\beta} A_{\bar\alpha})\Big]$$

I use overbars on indices here to explicitly distinguish which are being summed over. For example, μ and $\bar\mu$ are distinct indices.

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{-1}{4}\Bigg[\frac{\partial(\partial_{\bar\mu} A_{\bar\nu})}{\partial(\partial_\nu A_\mu)}\left(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \partial_{\bar\alpha} A_{\bar\beta}\right) + \left(\partial_{\bar\mu} A_{\bar\nu}\right)\left(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \frac{\partial(\partial_{\bar\alpha} A_{\bar\beta})}{\partial(\partial_\nu A_\mu)}\right)$$

$$- \frac{\partial(\partial_{\bar\mu} A_{\bar\nu})}{\partial(\partial_\nu A_\mu)}\left(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \partial_{\bar\beta} A_{\bar\alpha}\right) - \left(\partial_{\bar\mu} A_{\bar\nu}\right)\left(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \frac{\partial(\partial_{\bar\beta} A_{\bar\alpha})}{\partial(\partial_\nu A_\mu)}\right)$$

$$- \frac{\partial(\partial_{\bar\nu} A_{\bar\mu})}{\partial(\partial_\nu A_\mu)}\left(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \partial_{\bar\alpha} A_{\bar\beta}\right) - \left(\partial_{\bar\nu} A_{\bar\mu}\right)\left(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \frac{\partial(\partial_{\bar\alpha} A_{\bar\beta})}{\partial(\partial_\nu A_\mu)}\right)$$

$$+ \frac{\partial(\partial_{\bar\nu} A_{\bar\mu})}{\partial(\partial_\nu A_\mu)}\left(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \partial_{\bar\beta} A_{\bar\alpha}\right) + \left(\partial_{\bar\nu} A_{\bar\mu}\right)\left(g^{\bar\alpha\bar\mu} g^{\bar\beta\bar\nu} \frac{\partial(\partial_{\bar\beta} A_{\bar\alpha})}{\partial(\partial_\nu A_\mu)}\right)\Bigg]$$

In each of these steps use $\frac{\partial x^\rho}{\partial x^\sigma} = \delta^\rho_\sigma$.

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{-1}{4}\Bigg[\frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\nu A_\mu)}\left(g^{\bar\alpha\nu} g^{\bar\beta\mu} \partial_{\bar\alpha} A_{\bar\beta}\right) + \left(\partial_{\bar\mu} A_{\bar\nu}\right)\left(g^{\nu\bar\mu} g^{\mu\bar\nu} \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\nu A_\mu)}\right)$$

$$- \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\nu A_\mu)}\left(g^{\bar\alpha\nu} g^{\bar\beta\mu} \partial_{\bar\beta} A_{\bar\alpha}\right) - \left(\partial_{\bar\mu} A_{\bar\nu}\right)\left(g^{\mu\bar\mu} g^{\nu\bar\nu} \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\nu A_\mu)}\right)$$

$$- \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\nu A_\mu)}\left(g^{\bar\alpha\mu} g^{\bar\beta\nu} \partial_{\bar\alpha} A_{\bar\beta}\right) - \left(\partial_{\bar\nu} A_{\bar\mu}\right)\left(g^{\nu\bar\mu} g^{\mu\bar\nu} \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\nu A_\mu)}\right)$$

$$+ \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\nu A_\mu)}\left(g^{\bar\alpha\mu} g^{\bar\beta\nu} \partial_{\bar\beta} A_{\bar\alpha}\right) + \left(\partial_{\bar\nu} A_{\bar\mu}\right)\left(g^{\mu\bar\mu} g^{\nu\bar\nu} \frac{\partial(\partial_\nu A_\mu)}{\partial(\partial_\nu A_\mu)}\right)\Bigg]$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{-1}{4}\Big[g^{\bar\alpha\nu} g^{\bar\beta\mu} \partial_{\bar\alpha} A_{\bar\beta} + \partial_{\bar\mu} A_{\bar\nu} g^{\nu\bar\mu} g^{\mu\bar\nu} - g^{\bar\alpha\nu} g^{\bar\beta\mu} \partial_{\bar\beta} A_{\bar\alpha} - \partial_{\bar\mu} A_{\bar\nu} g^{\mu\bar\mu} g^{\nu\bar\nu}$$

$$- g^{\bar\alpha\mu} g^{\bar\beta\nu} \partial_{\bar\alpha} A_{\bar\beta} - \partial_{\bar\nu} A_{\bar\mu} g^{\nu\bar\mu} g^{\mu\bar\nu} + g^{\bar\alpha\mu} g^{\bar\beta\nu} \partial_{\bar\beta} A_{\bar\alpha} + \partial_{\bar\nu} A_{\bar\mu} g^{\mu\bar\mu} g^{\nu\bar\nu} \Big]$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{-1}{4}\Big[\partial^\nu A^\mu + \partial^\nu A^\mu - \partial^\mu A^\nu - \partial^\mu A^\nu - \partial^\mu A^\nu - \partial^\mu A^\nu + \partial^\nu A^\mu + \partial^\nu A^\mu \Big]$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \frac{-1}{\cancel{4}}\Big[\cancel{4}\partial^\nu A^\mu - \cancel{4}\partial^\mu A^\nu \Big]$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu}$$

Putting the Euler-Lagrange equation back together, the equation of motion is

$$\boxed{\partial_\nu F^{\mu\nu} = 0}$$

Using $\partial_\mu = (\partial_0, \partial_i)$, $i = 1,2,3$

$$\Rightarrow \partial_0 F^{\mu 0} + \partial_i F^{\mu i} = 0$$

and for $\mu = 0$, $F^{00} = 0$,

$$\Rightarrow \partial_i F^{0i} = 0$$

$$\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = 0}$$

where $E^i = -F^{0i}$.

For $\mu = j$, $j = 1,2,3$,

$$\Rightarrow \partial_0 F^{j0} + \partial_i F^{ji} = 0$$

$$\Rightarrow \partial_0 F^{j0} + \partial_i (-\varepsilon^{jik} B^k) = 0$$

$$\Rightarrow -\frac{\partial \vec{E}}{\partial t} + \partial_i (\varepsilon^{ijk} B^k) = 0$$

$$\Rightarrow \boxed{-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = 0}$$

There we have two of Maxwell's Equations. The other two are derived in a similar fashion from the ~~of~~ other Euler-Lagrange equation (II):

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial^\nu \left[\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} \right] = 0$$

This time, I write the Lagrangian density with all raised indices.

$$\mathcal{L} = \frac{-1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$= \frac{-1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= \frac{-1}{4} \left[(g_{\alpha\mu} g_{\beta\nu} \partial^\alpha A^\beta - g_{\alpha\mu} g_{\beta\nu} \partial^\beta A^\alpha)(\partial^\mu A^\nu - \partial^\nu A^\mu) \right]$$

$$= \frac{-1}{4} \Big[(g_{\alpha\mu} g_{\beta\nu} \partial^\alpha A^\beta)(\partial^\mu A^\nu) - (g_{\alpha\mu} g_{\beta\nu} \partial^\alpha A^\beta)(\partial^\nu A^\mu)$$

$$- (g_{\alpha\mu} g_{\beta\nu} \partial^\beta A^\alpha)(\partial^\mu A^\nu) + (g_{\alpha\mu} g_{\beta\nu} \partial^\beta A^\alpha)(\partial^\nu A^\mu) \Big]$$

Again, there's the trivial case.

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = \left(\frac{-1}{4}\right)\Bigg[\left(g_{\bar\alpha\bar\mu}\, g_{\bar\beta\bar\nu}\, \frac{\partial(\partial^{\bar\alpha} A^{\bar\beta})}{\partial(\partial^\nu A^\mu)}\right)\left(\partial^{\bar\mu} A^{\bar\nu}\right) + \left(g_{\bar\alpha\bar\mu}\, g_{\bar\beta\bar\nu}\, \partial^{\bar\alpha} A^{\bar\beta}\right)\left(\frac{\partial(\partial^{\bar\mu} A^{\bar\nu})}{\partial(\partial^\nu A^\mu)}\right)$$

$$-\left(g_{\bar\alpha\bar\mu}\, g_{\bar\beta\bar\nu}\, \frac{\partial(\partial^{\bar\alpha} A^{\bar\beta})}{\partial(\partial^\nu A^\mu)}\right)\partial^{\bar\nu} A^{\bar\mu} - \left(g_{\bar\alpha\bar\mu}\, g_{\bar\beta\bar\nu}\, \partial^{\bar\alpha} A^{\bar\beta}\right)\frac{\partial(\partial^{\bar\nu} A^{\bar\mu})}{\partial(\partial^\nu A^\mu)}$$

$$-\left(g_{\bar\alpha\bar\mu}\, g_{\bar\beta\bar\nu}\, \frac{\partial(\partial^{\bar\beta} A^{\bar\alpha})}{\partial(\partial^\nu A^\mu)}\right)\partial^{\bar\mu} A^{\bar\nu} - \left(g_{\bar\alpha\bar\mu}\, g_{\bar\beta\bar\nu}\, \partial^{\bar\beta} A^{\bar\alpha}\right)\frac{\partial(\partial^{\bar\mu} A^{\bar\nu})}{\partial(\partial^\nu A^\mu)}$$

$$+\left(g_{\bar\alpha\bar\mu}\, g_{\bar\beta\bar\nu}\, \frac{\partial(\partial^{\bar\beta} A^{\bar\alpha})}{\partial(\partial^\nu A^\mu)}\right)\partial^{\bar\nu} A^{\bar\mu} + \left(g_{\bar\alpha\bar\mu}\, g_{\bar\beta\bar\nu}\, \partial^{\bar\beta} A^{\bar\alpha}\right)\frac{\partial(\partial^{\bar\nu} A^{\bar\mu})}{\partial(\partial^\nu A^\mu)}\Bigg]$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = \left(\frac{-1}{4}\right)\Big[\, g_{\nu\bar\mu}\, g_{\mu\bar\nu}\, \partial^{\bar\mu} A^{\bar\nu} + g_{\bar\alpha\nu}\, g_{\bar\beta\mu}\, \partial^{\bar\alpha} A^{\bar\beta} - g_{\nu\bar\mu}\, g_{\mu\bar\nu}\, \partial^{\bar\nu} A^{\bar\mu} - g_{\bar\alpha\mu}\, g_{\bar\beta\nu}\, \partial^{\bar\alpha} A^{\bar\beta}$$

$$- g_{\mu\bar\mu}\, g_{\nu\bar\nu}\, \partial^{\bar\mu} A^{\bar\nu} - g_{\bar\alpha\nu}\, g_{\bar\beta\mu}\, \partial^{\bar\beta} A^{\bar\alpha} + g_{\mu\bar\mu}\, g_{\nu\bar\nu}\, \partial^{\bar\nu} A^{\bar\mu} + g_{\bar\alpha\mu}\, g_{\bar\beta\nu}\, \partial^{\bar\beta} A^{\bar\alpha}\Big]$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = \left(\frac{-1}{4}\right)\Big[\, \partial_\nu A_\mu + \partial_\nu A_\mu - \partial_\mu A_\nu - \partial_\mu A_\nu - \partial_\mu A_\nu - \partial_\mu A_\nu + \partial_\nu A_\mu + \partial_\nu A_\mu \Big]$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = \left(\frac{-1}{4}\right)\Big[\, 4\,\partial_\nu A_\mu - 4\,\partial_\mu A_\nu \Big]$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial^\nu A^\mu)} = F_{\mu\nu}$$

Putting the pieces of the Euler-Lagrange Eqn. back together:

$$\boxed{\partial^\nu F_{\mu\nu} = 0}$$

We can put this into a form more similar to the $\partial_\nu F^{\mu\nu} = 0$ equation by defining the "dual field strength tensor" $\mathcal{F}^{\rho\sigma} \equiv \frac{1}{2}\varepsilon^{\rho\sigma\mu\nu} F_{\mu\nu}$ (Jackson, 3rd ed., p. 556).

So, as the right-hand side is zero

$$\Rightarrow \quad g_{\rho\nu}\partial^\nu\left(\frac{1}{2}\varepsilon^{\rho\sigma\mu\nu} F_{\mu\nu}\right) = 0$$

$$\Rightarrow \quad \boxed{\partial_\rho \mathcal{F}^{\rho\sigma} = 0}$$

Correspondingly, $B^i = -\mathcal{F}^{0i}$ and $\varepsilon^{ijk}E^k = -\mathcal{F}^{ij}$.

Thus,

$$\partial_0 \mathcal{F}^{0\sigma} + \partial_i \mathcal{F}^{i\sigma} = 0$$

$$\partial_i \mathcal{F}^{i0} = 0$$

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0} \quad \text{for } \sigma = 0.$$

And, finally, for $\sigma = j$; $j = 1, 2, 3$,

$$\Rightarrow \partial_0 \mathcal{F}^{0j} + \partial_i \mathcal{F}^{ij} = 0$$

$$\Rightarrow -\frac{\partial \vec{B}}{\partial t} - \partial_i \varepsilon^{ijk} E^k = 0$$

$$\Rightarrow \boxed{\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0}$$

Here, we have the last of Maxwell's equations.

2b) In classical mechanics for particles, the Hamiltonian is

$$H = \sum_i p_i \dot{q}_i - L$$

where L is the Lagrangian and

$$p_i = \frac{\partial L}{\partial \dot{q}_i} .$$

For fields we have a 'Lagrangian density' and, thus, an analogous 'Hamiltonian density' defined in a fashion similar to what's above:

$$\mathcal{H} \equiv \sum_k \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi_k}{\partial t}\right)} \frac{\partial \phi_k}{\partial t} - \mathcal{L}$$

If the Hamiltonian density is to be relativistically covariant, it must be one part of a more general quantity called the "<u>canonical stress tensor</u>" or "<u>energy-momentum tensor</u>." The Hamiltonian would be the energy part of it.

So, generalizing the form of the Hamiltonian density to all 4 spacetime components:

$$T^{\alpha\beta} = \sum_k \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta \phi_k - g^{\alpha\beta} \mathcal{L}$$

Inputting the free EM Lagrangian in particular:

$$T^{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^\mu)} \partial^\beta A^\mu - g^{\alpha\beta}\mathcal{L}$$

but I calculated $\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A^\mu)}$ in 2.1(a), so

$$\boxed{T^{\alpha\beta} = g^{\alpha\nu} F_{\nu\lambda} (\partial^\beta A^\lambda) - g^{\alpha\beta}\mathcal{L}}$$

where

$$\mathcal{L} = \frac{-1}{4}\left[F_{\mu 0} F^{\mu 0} + F_{\mu 1} F^{\mu 1} + F_{\mu 2} F^{\mu 2} + F_{\mu 3} F^{\mu 3}\right]$$

$$\mathcal{L} = \frac{-1}{4}\left[-|\vec{E}|^2 - |\vec{E}|^2 + 2|\vec{B}|^2\right] \qquad (\text{using } F^{00} = F_{00} = 0)$$

$$\boxed{\mathcal{L} = \frac{1}{2}\left[|\vec{E}|^2 - |\vec{B}|^2\right]}$$

However, $T^{\alpha\beta} \neq T^{\beta\alpha}$ so the Energy-Momentum Tensor is not symmetric. The technique to make it symmetric (or have an analogous symmetric version) is to introduce $K^{\bar{\lambda}\alpha\beta} \equiv F^{\alpha\bar{\lambda}} A^\beta$, and define

$$\hat{T}^{\alpha\beta} \equiv T^{\alpha\beta} - \partial_{\bar{\lambda}}\left(K^{\bar{\lambda}\alpha\beta}\right)$$

$$\Rightarrow \hat{T}^{\alpha\beta} = T^{\alpha\beta} - \partial_{\bar{\lambda}}(F^{\alpha\bar{\lambda}} A^{\beta})$$

$$= T^{\alpha\beta} - (\cancelto{0}{\partial_{\bar{\lambda}} F^{\alpha\bar{\lambda}}}) A^{\beta} - F^{\alpha\bar{\lambda}}(\partial_{\bar{\lambda}} A^{\beta})$$

We know that $\partial_{\lambda} F^{\alpha\bar{\lambda}} = 0$ from earlier in this problem; it contained two of Maxwell's equations.

$$\hat{T}^{\alpha\beta} = g^{\alpha\nu} F_{\nu\lambda} (\partial^{\beta} A^{\lambda}) - F^{\alpha\bar{\lambda}}(\partial_{\bar{\lambda}} A^{\beta}) - g^{\alpha\beta} \mathcal{L}$$

$$\hat{T}^{\alpha\beta} = g^{\alpha\nu} F_{\nu\lambda} g^{\bar{\lambda}\lambda} (\partial^{\beta} A_{\bar{\lambda}}) - F^{\alpha\bar{\lambda}}(\partial_{\bar{\lambda}} A^{\beta}) - g^{\alpha\beta} \mathcal{L}$$

$$\hat{T}^{\alpha\beta} = F^{\alpha\bar{\lambda}} (\partial^{\beta} A_{\bar{\lambda}}) - F^{\alpha\bar{\lambda}}(\partial_{\bar{\lambda}} A^{\beta}) - g^{\alpha\beta} \mathcal{L}$$

$$\hat{T}^{\alpha\beta} = F^{\alpha\bar{\lambda}} (\partial^{\beta} A_{\bar{\lambda}} - \partial_{\bar{\lambda}} A^{\beta}) - g^{\alpha\beta} \mathcal{L}$$

$$\boxed{\hat{T}^{\alpha\beta} = F^{\alpha\bar{\lambda}} F^{\beta}{}_{\bar{\lambda}} - g^{\alpha\beta} \mathcal{L}}$$