The Photon Propagator

This is an exercise in how gauge theories are a necessary part of particle physics. In the next few pages we will see how the covariant expression of the Maxwell equations must be altered by choice of gauge in order to calculate the photon propagator, $\langle 0|T[A^\nu(x_1)A^\nu(x_2)]|0\rangle$.

The Maxwell Lagrangian is

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

and the resulting covariant Maxwell equations of motion are

$$\Box A^\nu - \partial^\nu \partial^\mu A_\mu = j^\nu_{em}$$

and for a free field, $j^\nu_{em} = 0$ so

$$\Box A^\nu - \partial^\nu \partial^\mu A_\mu = 0$$

Assume the plane wave mode expansion for A_μ.

$$A_\mu = \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega}} \left[\epsilon_\mu(k,\lambda)\, a(k,\lambda)\, e^{ik\cdot x} + \epsilon^*_\mu(k,\lambda)\, a(k,\lambda)\, e^{-ik\cdot x}\right]$$

Put A^ν, A_μ into Maxwell's equation.

$$\Rightarrow \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega}} \left[\epsilon_\mu(k,\lambda)\, a(k,\lambda)\, \Box e^{ik\cdot x} + \epsilon^*_\mu(k,\lambda)\, a^*(k,\lambda)\, \Box e^{-ik\cdot x} \right]$$

$$- \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega}} \left[\epsilon_\mu(k,\lambda)\, a(k,\lambda)\, \partial^\nu \partial_\nu e^{ik\cdot x} + \epsilon^*_\mu(k,\lambda)\, a^*(k,\lambda)\, \partial^\nu \partial_\nu e^{-ik\cdot x} \right] = 0$$

$$\Rightarrow \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega}} \left[\epsilon_\mu(k,\lambda)\, a(k,\lambda)(-k^2) e^{ik\cdot x} + \epsilon^*_\mu(k,\lambda)\, a^*(k,\lambda)(-k^2) e^{-ik\cdot x} \right]$$

$$- \sum_\lambda \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2\omega}} \left[\epsilon_\mu(k,\lambda)\, a(k,\lambda)(-k^\nu k_\nu) e^{ik\cdot x} + \epsilon^*_\mu(k,\lambda)\, a^*(k,\lambda)(-k^\nu k_\nu) e^{-ik\cdot x} \right] = 0$$

$$\Rightarrow \left(-k^2 g^{\mu\nu} + k^\nu k^\mu\right) \tilde{A}_\mu(k) = 0 \qquad \text{eqn. (7.87)}$$

Define $M^{\nu\mu} \equiv -k^2 g^{\nu\mu} + k^\nu k^\mu$. The inverse of $M^{\nu\mu}$ is suppose to yield the propagator, but it does not exist. Try assuming a more general form of the inverse.

$$(M^{-1})^{\nu\mu} = A(k^2)\, g^{\nu\mu} + B(k^2)\, k^\nu k^\mu$$

where $A(k^2)$ and $B(k^2)$ are arbitrary functions. As a Lorentz covariant 4-tensor, the inverse is defined by the condition

$$(M^{-1})^{\nu\mu} M_{\mu\sigma} = g^\nu_\sigma$$

$$\Rightarrow \left[A(k^2)g^{\mu\nu} + B(k^2)k^\nu k^\nu\right]\left[-k^2 g_{\mu\sigma} + k_\mu k_\sigma\right] = g^\nu_\sigma$$

$$-k^2 A(k^2) g^{\nu\mu} g_{\mu\sigma} + A(k^2) g^{\nu\mu} k_\mu k_\sigma - k^2 B(k^2) k^\nu k^\mu g_{\mu\sigma} + B(k^2) k^\nu k^\mu k_\mu k_\sigma = g^\nu_\sigma$$

$$-k^2 A(k^2) g^\nu_\sigma + A(k^2) k^\nu k_\sigma - k^2 B(k^2) k^\nu k_\sigma + B(k^2) k^\nu k^2 k_\sigma = g^\nu_\sigma$$

$$-k^2 A(k^2) g^\nu_\sigma + A(k^2) k^\nu k_\sigma = g^\nu_\sigma$$

An inverse does not exist even for this more general form.

How do we deal with this problem?

Recall that we started with

$$\Box A^\nu - \partial^\nu \partial^\mu A_\mu = 0$$

This time, we first perform a gauge transformation

$$A^\nu \rightarrow A^\nu - \partial^\nu \chi$$

The Maxwell equations are invariant under such a transformation. As such we can choose A^ν such that the Lorentz condition is satisfied

$$\partial_\mu A^\nu = 0 .$$

The value in this choice of gauge is that the components of A^ν decouple in a relativistically covariant manner, as indicated by the repeated "μ" index.

Anyhow, we begin this time with

$$\Box A^\nu = 0$$

instead. Again, using the mode expansion for A^ν

$$\Rightarrow -k^2 \tilde{A}^\nu = 0$$

We see that $-k^2$ certainly does have an inverse, as hoped for.

The Lagrangian that produces $\Box A^\nu = 0$ is

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2$$

and the resulting commutation relations that describe the nature of the quantization are

$$[A_\mu(\vec{x},t), A_\nu(\vec{y},t)] = 0$$

$$[A_\mu(\vec{x},t), \dot{A}_\nu(\vec{y},t)] = -i g_{\mu\nu} \delta^3(\vec{x}-\vec{y})$$

$$[a_\lambda(\vec{k}), a^\dagger_{\lambda'}(\vec{k}')] = -g_{\lambda\lambda'} (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

Using these relations, we can solve for the Feynman photon propagator

$$\langle 0|T[A^\mu(x_1)A^\nu(x_2)]|0\rangle = \langle 0|\underbrace{[\Theta(t_1-t_2)A^\mu(x_1)A^\nu(x_2)}_{\text{for } t_1 > t_2} + \underbrace{\Theta(t_2-t_1)A^\nu(x_2)A^\mu(x_1)]}_{\text{for } t_2 > t_1}|0\rangle$$

$$= \langle 0 | \overleftarrow{T}(t_1 - t_2) \sum_{\lambda_1=0}^{3} \int \frac{d^3\vec{k}_1}{(2\pi)^3 \sqrt{2\omega_1}} \left[\epsilon^{\mu}(k_1,\lambda_1)\, a_{\lambda_1}(k_1)\, e^{-ik_1\cdot x_1} + \epsilon^{*\mu}(k_1,\lambda_1)\, a^{\dagger}_{\lambda_1}(k_1)\, e^{ik_1\cdot x_1} \right]$$

$$\cdot \sum_{\lambda_2=0}^{3} \int \frac{d^3\vec{k}_2}{(2\pi)^3 \sqrt{2\omega_2}} \left[\epsilon^{\nu}(k_2,\lambda_2)\, a_{\lambda_2}(k_2)\, e^{-ik_2\cdot x_2} + \epsilon^{*\nu}(k_2,\lambda_2)\, a^{\dagger}_{\lambda_2}(k_2)\, e^{ik_2\cdot x_2} \right] | 0 \rangle$$

$$+ \langle 0 | \overleftarrow{T}(t_2 - t_1) \sum_{\lambda_2=0}^{3} \int \frac{d^3\vec{k}_2}{(2\pi)^3 \sqrt{2\omega_2}} \left[\epsilon^{\nu}(k_2,\lambda_2)\, a_{\lambda_2}(k_2)\, e^{-ik_2\cdot x_2} + \epsilon^{*\nu}(k_2,\lambda_2)\, a^{\dagger}_{\lambda_2}(k_2)\, e^{ik_2\cdot x_2} \right]$$

$$\cdot \sum_{\lambda_1=0}^{3} \int \frac{d^3\vec{k}_1}{(2\pi)^3 \sqrt{2\omega_1}} \left[\epsilon^{\mu}(k_1,\lambda_1)\, a_{\lambda_1}(k_1)\, e^{-ik_1\cdot x_1} + \epsilon^{*\mu}(k_1,\lambda_1)\, a^{\dagger}_{\lambda_1}(k_1)\, e^{ik_1\cdot x_1} \right] | 0 \rangle$$

$$= \langle 0 | \overleftarrow{T}(t_1 - t_2) \sum_{\lambda_1,\lambda_2} \iint \frac{d^3\vec{k}_1\, d^3\vec{k}_2}{(2\pi)^6\, 2\sqrt{\omega_1\omega_2}} \Big[\epsilon^{\mu}(k_1,\lambda_1)\, a_{\lambda_1}(k_1)\, \epsilon^{\nu}(k_2,\lambda_2)\, a_{\lambda_2}(k_2)\, e^{-ik_1\cdot x_1}\, e^{-ik_2\cdot x_2}$$

$$+ \epsilon^{\mu}(k_1,\lambda_1)\, a_{\lambda_1}(k_1)\, \epsilon^{*\nu}(k_2,\lambda_2)\, a^{\dagger}_{\lambda_2}(k_2)\, e^{-ik_1\cdot x_1}\, e^{ik_2\cdot x_2}$$

$$+ \epsilon^{*\mu}(k_1,\lambda_1)\, a^{\dagger}_{\lambda_1}(k_1)\, \epsilon^{\nu}(k_2,\lambda_2)\, a_{\lambda_2}(k_2)\, e^{ik_1\cdot x_1}\, e^{-ik_2\cdot x_2}$$

$$+ \epsilon^{*\mu}(k_1,\lambda_1)\, a^{\dagger}_{\lambda_1}(k_1)\, \epsilon^{*\nu}(k_2,\lambda_2)\, a^{\dagger}_{\lambda_2}(k_2)\, e^{ik_1\cdot x_1}\, e^{ik_2\cdot x_2} \Big] | 0 \rangle$$

$$+ \langle 0 | \overleftarrow{T}(t_2 - t_1) \sum_{\lambda_1,\lambda_2} \iint \frac{d^3\vec{k}_2\, d^3\vec{k}_1}{(2\pi)^6\, 2\sqrt{\omega_1\omega_2}} \Big[\epsilon^{\nu}(k_2,\lambda_2)\, a_{\lambda_2}(k_2)\, \epsilon^{\mu}(k_1,\lambda_1)\, a_{\lambda_1}(k_1)\, e^{-ik_2\cdot x_2}\, e^{-ik_1\cdot x_1}$$

$$+ \epsilon^{\nu}(k_2,\lambda_2)\, a_{\lambda_2}(k_2)\, \epsilon^{*\mu}(k_1,\lambda_1)\, a^{\dagger}_{\lambda_1}(k_1)\, e^{-ik_2\cdot x_2}\, e^{ik_1\cdot x_1}$$

$$+ \epsilon^{*\nu}(k_2,\lambda_2)\, a^{\dagger}_{\lambda_2}(k_2)\, \epsilon^{\mu}(k_1,\lambda_1)\, a_{\lambda_1}(k_1)\, e^{ik_2\cdot x_2}\, e^{-ik_1\cdot x_1}$$

$$+ \epsilon^{*\nu}(k_2,\lambda_2)\, a^{\dagger}_{\lambda_2}(k_2)\, \epsilon^{*\mu}(k_1,\lambda_1)\, a^{\dagger}_{\lambda_1}(k_1)\, e^{ik_2\cdot x_2}\, e^{ik_1\cdot x_1} \Big] | 0 \rangle$$

$$= \langle 0 | \theta(t_1 - t_2) \sum_{\lambda_1,\lambda_2} \iint \frac{d^3\vec{k}_1\, d^3\vec{k}_2}{(2\pi)^6\, 2\sqrt{\omega_1\omega_2}} \Big[\epsilon^{\mu}(k_1,\lambda_1)\alpha_{\lambda_1}(k_1)\epsilon^{*\nu}(k_2,\lambda_2)\alpha^{\dagger}_{\lambda_2}(k_2) e^{-ik_1\cdot x_1} e^{ik_2\cdot x_2} \Big] | 0 \rangle$$

$$+ \langle 0 | \theta(t_2 - t_1) \sum_{\lambda_1,\lambda_2} \iint \frac{d^3\vec{k}_2\, d^3\vec{k}_1}{(2\pi)^6\, 2\sqrt{\omega_2\omega_1}} \Big[\epsilon^{\nu}(k_2,\lambda_2)\alpha_{\lambda_2}(k_2)\epsilon^{*\mu}(k_1,\lambda_1)\alpha^{\dagger}_{\lambda_1}(k_1) e^{-ik_2\cdot x_2} e^{ik_1\cdot x_1} \Big] | 0 \rangle$$

$$= \langle 0 | \theta(t_1 - t_2) \sum_{\lambda_1,\lambda_2} \iint \frac{d^3\vec{k}_1\, d^3\vec{k}_2}{(2\pi)^6\, 2\sqrt{\omega_1\omega_2}} \Big[\epsilon^{\mu}(k_1,\lambda_1)\epsilon^{*\nu}(k_2,\lambda_2)\alpha_{\lambda_1}(k_1)\alpha^{\dagger}_{\lambda_2}(k_2) e^{-ik_1\cdot x_1} e^{ik_2\cdot x_2} \Big] | 0 \rangle$$

$$+ \langle 0 | \theta(t_2 - t_1) \sum_{\lambda_1,\lambda_2} \iint \frac{d^3\vec{k}_2\, d^3\vec{k}_1}{(2\pi)^6\, 2\sqrt{\omega_2\omega_1}} \Big[\epsilon^{\nu}(k_2,\lambda_2)\epsilon^{*\mu}(k_1,\lambda_1)\alpha_{\lambda_2}(k_2)\alpha^{\dagger}_{\lambda_1}(k_1) e^{-ik_2\cdot x_2} e^{ik_1\cdot x_1} \Big] | 0 \rangle$$

Now we can use the following commutation relations:

$$\Big[\alpha_{\lambda_2}(k_2),\, \alpha^{\dagger}_{\lambda_1}(k_1) \Big] = -g_{\lambda_2\lambda_1} (2\pi)^3 \delta^3(\vec{k}_2 - \vec{k}_1)$$

$$\alpha_{\lambda_2}(k_2)\alpha^{\dagger}_{\lambda_1}(k_1) = -g_{\lambda_2\lambda_1}(2\pi)^3\delta^3(\vec{k}_2 - \vec{k}_1) + \alpha^{\dagger}_{\lambda_1}(k_1)\alpha_{\lambda_2}(k_2)$$

and

$$\Big[\alpha_{\lambda_1}(k_1),\, \alpha^{\dagger}_{\lambda_2}(k_2) \Big] = -g_{\lambda_1\lambda_2} (2\pi)^3 \delta^3(\vec{k}_1 - \vec{k}_2)$$

$$\alpha_{\lambda_1}(k_1)\alpha^{\dagger}_{\lambda_2}(k_2) = -g_{\lambda_1\lambda_2}(2\pi)^3\delta^3(\vec{k}_1 - \vec{k}_2) + \alpha^{\dagger}_{\lambda_2}(k_2)\alpha_{\lambda_1}(k_1)$$

Back to the propagator:

$$\langle 0 | T[A^{\mu}(x_1) A^{\nu}(x_2)] | 0 \rangle$$

$$= -\langle 0 | \theta(t_1 - t_2) \sum_{\lambda_1,\lambda_2} \iint \frac{d^3\vec{k}_1\, d^3\vec{k}_2}{(2\pi)^6\, 2\sqrt{\omega_1\omega_2}} \epsilon^{\mu}(k_1,\lambda_1)\epsilon^{*\nu}(k_2,\lambda_2) g_{\lambda_1\lambda_2} (2\pi)^3\delta^3(\vec{k}_1 - \vec{k}_2) e^{-ik_1\cdot x_1} e^{ik_2\cdot x_2} | 0 \rangle$$

$$- \langle 0 | \theta(t_2 - t_1) \sum_{\lambda_1,\lambda_2} \iint \frac{d^3\vec{k}_2\, d^3\vec{k}_1}{(2\pi)^6\, 2\sqrt{\omega_2\omega_1}} \epsilon^{\nu}(k_2,\lambda_2)\epsilon^{*\mu}(k_1,\lambda_1) g_{\lambda_2\lambda_1} (2\pi)^3\delta^3(\vec{k}_2 - \vec{k}_1) e^{-ik_2\cdot x_2} e^{ik_1\cdot x_1} | 0 \rangle$$

Integrate over k_2.

$$= -\langle 0|\Theta(t_1-t_2)\sum_{\lambda_1,\lambda_2}\int\frac{d^3\vec{k}_1}{(2\pi)^3 2\omega_1}\epsilon^{\mu}(k_1,\lambda_1)\epsilon^{*\nu}(k_1,\lambda_2)g_{\lambda_1\lambda_2}e^{ik_1\cdot(x_1-x_2)}|0\rangle$$

$$-\langle 0|\Theta(t_2-t_1)\sum_{\lambda_2,\lambda_1}\int\frac{d^3\vec{k}_1}{(2\pi)^3 2\omega_1}\epsilon^{\nu}(k_1,\lambda_2)\epsilon^{*\mu}(k_1,\lambda_1)g_{\lambda_2\lambda_1}e^{-ik_1\cdot(x_2-x_1)}|0\rangle$$

We need a quick digression to see how the sums are dealt with. Recall that

$$g_{\mu\nu}=\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{bmatrix}$$

$\epsilon^{\mu}(\lambda=0)=[1,0,0,0]$ "time-like polarization"

$\epsilon^{\mu}(\lambda=1)=[0,1,0,0]$ } "Transverse polarization"

$\epsilon^{\mu}(\lambda=2)=[0,0,1,0]$ }

$\epsilon^{\mu}(\lambda=3)=[0,0,0,1]$ "Longitudinal polarization"

Note how each of the four polarizations are orthogonal. Because of this, summing over λ_2 in something like $\epsilon^{\mu}(k_1,\lambda_1)\epsilon^{*\nu}(k_1,\lambda_2)$ will yield only one nonvanishing term, the term corresponding to $\lambda_1=\lambda_2$. Hence, $\epsilon^{\mu}(k_1,\lambda_1)\epsilon^{*\nu}(k_1,\lambda_2)=\epsilon^{\mu}(k_1,\lambda_1)\epsilon^{*\nu}(k_1,\lambda_1)=1$.

Sum over λ_2.

$$= -\langle 0 | \mathcal{T}(x_1 - x_2) \sum_{\lambda=0}^{3} \int \frac{d^3\vec{k}_1}{(2\pi)^3 \sqrt{2\omega_1}} \epsilon^{\mu}(k_1, \lambda_1) \epsilon^{*\nu}(k_1, \lambda_1) g_{\lambda_1 \lambda_1} e^{-ik_1 \cdot (x_1 - x_2)} | 0 \rangle$$

$$- \langle 0 | \mathcal{T}(x_2 - x_1) \sum_{\lambda=0}^{3} \int \frac{d^3\vec{k}_1}{(2\pi)^3 \sqrt{2\omega_1}} \epsilon^{\nu}(k_1, \lambda_1) \epsilon^{*\mu}(k_1, \lambda_1) g_{\lambda_1 \lambda_1} e^{-ik_1 \cdot (x_2 - x_1)} | 0 \rangle$$

In performing the sum over λ_1, we need to utilize the "pseudo-completeness" relation, eqn. (7.115).

$$\sum_{\lambda=0}^{3} \epsilon^{\mu}(k_1, \lambda_1) \epsilon^{\nu}(k_1, \lambda_1) g_{\lambda_1 \lambda_1}$$

$$= -\epsilon^{\mu}(k_1, 0) \epsilon^{\nu}(k_1, 0) + \epsilon^{\mu}(k_1, 1) \epsilon^{\nu}(k_1, 1)$$

$$+ \epsilon^{\mu}(k_1, 2) \epsilon^{\nu}(k_1, 2) + \epsilon^{\mu}(k_1, 3) \epsilon^{\nu}(k_1, 3)$$

$$= -g^{\mu\nu}.$$

Sum over λ_1.

$$= \langle 0 | \mathcal{T}(x_1 - x_2) \int \frac{d^3k}{(2\pi)^3 2\omega} g^{\mu\nu} e^{-ik \cdot (x_1 - x_2)} | 0 \rangle$$

$$+ \langle 0 | \mathcal{T}(x_2 - x_1) \int \frac{d^3k}{(2\pi)^3 2\omega} g^{\mu\nu} e^{-ik \cdot (x_2 - x_1)} | 0 \rangle$$

I dropped the subscript on $k_1 = k$ b/c it's no longer necessary.

$$= g^{\mu\nu} \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega} \left[\theta(t_1 - t_2) e^{-ik\cdot(x_1 - x_2)} + \theta(t_2 - t_1) e^{-ik\cdot(x_2 - x_1)} \right]$$

Aside from the $g^{\mu\nu}$, this is exactly the "Feynman Propagator in ABC Theory", so see that tag in this binder. Also, note that for photons $m = 0$.

$$\langle 0 | T[A^\mu(x_1) A^\nu(x_2)] | 0 \rangle$$

$$= g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} e^{-ik(x_1 - x_2)} \frac{i}{k_0^2 - \vec{k}^2 - m^2 + i\varepsilon} \qquad (\varepsilon \to 0)$$

$$= i g^{\mu\nu} \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot(x_1 - x_2)} \frac{1}{k^2 + i\varepsilon}$$

$$= \boxed{\frac{-i g^{\mu\nu}}{k^2 + i\varepsilon}}$$

The last step is just like problem 6.6 in solving for eqn. (6.100).