2a) Derive the equation of motion resulting from the Lagrangian density

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x)$$

To do this we use the Euler-Lagrange equation for fields.

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right] = 0 \\ \frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \right] = 0 \end{cases}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \frac{\partial}{\partial \bar{\psi}} \left[\bar{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) - \bar{\psi}(x) m \psi(x) \right] = i\gamma^\mu \partial_\mu \psi(x) - m\psi($$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \left[\bar{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) - \bar{\psi}(x) m \psi(x) \right] = 0$$

$$\Rightarrow i\gamma^\mu \partial_\mu \psi(x) - m\psi(x) - 0 = 0$$

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi(x) = 0}$$

Similarly, the Hermitian conjugate form is

$$\frac{\partial \mathcal{L}}{\partial \psi} = \frac{\partial}{\partial \psi} \left[\bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x) \right]$$

$$= \frac{\partial}{\partial \psi} \left[\bar{\psi}(x) i\gamma^\mu \partial_\mu \psi(x) - \bar{\psi}(x) m \psi(x) \right]$$

$$= -\bar{\psi}(x)\, m$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{\partial}{\partial(\partial_\mu \psi)}\left[\bar{\psi}(x) i\gamma^\nu \partial_\mu \psi(x) - \bar{\psi}(x) m \psi(x)\right]$$

$$= i\bar{\psi}(x)\gamma^\mu$$

$$\partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right] = i\partial_\mu \bar{\psi}(x)\gamma^\mu$$

$$\Rightarrow \boxed{-\bar{\psi}(x) m - i\partial_\mu(\bar{\psi}(x)\gamma^\mu) = 0}$$

2b) Note that the Dirac Lagrangian is invariant under

$$\psi \longrightarrow \psi' = e^{-i\alpha}\psi$$

and that this transformation necessarily implies

$$\bar{\psi} \longrightarrow \bar{\psi}' = e^{i\alpha}\bar{\psi}$$

So, substituting them into the Dirac Lagrangian

$$\mathcal{L} = e^{i\alpha}\bar{\psi}(i\gamma^\mu\partial_\mu - m)e^{-i\alpha}\psi$$

$$\Rightarrow \mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

Hence, the Lagrangian density is invariant under this transformation. It is also worth noting that $e^{i\alpha}$ and $e^{-i\alpha}$ may be brought together to cancel because the transformation is <u>global</u>; the 4-gradient ∂_μ therefore does not act on $e^{-i\alpha}$, which would otherwise lead to additional conditions.

The Dirac Lagrangian density is an explicit function of $\bar{\psi}(x)$, $\psi(x)$, and $\partial_\mu \psi(x)$. Let $\delta\mathcal{L}$ be the change in value of the Lagrangian density resulting from a transformation. As the particular transformation we are interested in ($\psi \to \psi' = e^{-i\alpha}\psi$) leaves $\mathcal{L}$ unchanged, we know $\delta\mathcal{L} = 0$. Thus, we may rewrite this quite generally as

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial\bar{\psi}}\delta\bar{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi) = 0$$

Use the Euler-Lagrange equation to make the following substitutions in the first two terms:

$$\delta\mathcal{L} = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right]\delta\psi + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right]\delta\bar{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi) = 0$$

And since $\delta(\partial_\mu\psi) = \partial_\mu(\delta\psi)$,

$$0 = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right]\delta\psi + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right]\delta\bar{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\partial_\mu(\delta\psi)$$

$$0 = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi\right] + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right]\delta\bar{\psi}$$

$$0 = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} \delta\bar{\psi} \right] \qquad (**)$$

Note that the steps taken so far are completely general. I have only used the knowledge that the Dirac Lagrangian density is a function of $\psi, \bar{\psi}$, and $\partial_\mu \psi$, but have not used the $\mathcal{L}$ itself.

First, the derivatives.

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}(x)\gamma^\mu$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0$$

Second, we need $\delta\psi$. To find this, let's consider the transformation in infinitesimal form.

$$\psi' = e^{-i\alpha}\psi$$

If $\alpha \to$ small, $e^{-i\alpha} \cong 1 - i\alpha$

$$\psi' \cong (1 - i\alpha)\psi$$

$$\psi' \cong \psi - i\alpha\psi$$

$$\delta\psi \equiv \psi' - \psi = i\alpha\psi$$

where we are now considering an infinitesimal transformation.

Now we are ready to substitute the particular terms associated with the Dirac Lagrangian density back into the general relation (**):

$$0 = \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi + \cancel{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}}^{0} \delta\bar{\psi} \right]$$

$$0 = \partial_\mu \left[i\bar{\psi}(x)\gamma^\mu (i\alpha\psi(x)) \right]$$

$$0 = \partial_\mu \left[\bar{\psi}(x)\gamma^\mu \psi(x) \right]$$

Finally, $\bar{\psi}(x)\gamma^\mu\psi(x) = \text{const.}$ is the physical quantity that is conserved under said transformation, otherwise denoted the "Noether Current".