The following is the derivation of the Hamiltonian resulting from the Dirac Lagrangian.

To begin, recall from classical field theory that the general form of a Hamiltonian is

$$H = \int d^3x \, \mathcal{H}$$

$$H = \int d^3x \left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial t}\right)} \frac{\partial \phi}{\partial t} - \mathcal{L} \right]$$

$$H = \int d^3x \left[\pi(\vec{x}) \dot{\phi}(\vec{x}) - \mathcal{L} \right]$$

where we have used the definition of the momentum field conjugate to $\phi(\vec{x})$: $\pi(\vec{x}) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$.

For Dirac theory in particular, the momentum field is

$$\pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}(\vec{x})} \qquad \mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

$$= \frac{\partial}{\partial \dot{\psi}(\vec{x})} \left[\bar{\psi}(i\gamma^\mu \partial_\mu \psi - m\psi) \right]$$

$$= \frac{\partial}{\partial \dot{\psi}(\vec{x})} \left[\bar{\psi}(i\gamma^0 \dot{\psi} + \vec{\gamma} \cdot \vec{\nabla} \psi - m\psi) \right]$$

$$= \frac{\partial}{\partial \dot{\psi}(\vec{x})} \left[i \psi^\dagger \gamma^0 \gamma^0 \dot{\psi} \right]$$

$$= \frac{\partial}{\partial \dot{\psi}(\vec{x})} \left[i\psi^\dagger \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \dot{\psi} \right]$$

$$= \frac{\partial}{\partial \dot{\psi}(\vec{x})} \left[i\psi^\dagger \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{\psi} \right]$$

$$= \frac{\partial}{\partial \dot{\psi}(\vec{x})} \left[i\psi^\dagger \dot{\psi} \right]$$

$$\underline{\pi(\vec{x}) = i\psi^\dagger}$$

Back to the Hamiltonian.

$$H = \int d^3x \left[i\psi^\dagger \dot{\psi}(\vec{x}) - \mathcal{L} \right]$$

$$H = \int d^3x \left[i\psi^\dagger \dot{\psi} - \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \right]$$

$$H = \int d^3x \left[i\psi^\dagger \dot{\psi} - \bar{\psi}(i\gamma^0 \partial_0 + i\vec{\gamma}\cdot\vec{\nabla} - m)\psi \right]$$

$$H = \int d^3x \left[i\psi^\dagger \dot{\psi} - i\psi^\dagger \gamma^0 \gamma^0 \dot{\psi} - i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi + \bar{\psi} m \psi \right]$$

$$H = \int d^3x \left[i\psi^\dagger \dot{\psi} - i\psi^\dagger \dot{\psi} - i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi + \bar{\psi} m \psi \right]$$

$$H = \int d^3x \left[-i\bar{\psi}(\vec{\gamma}\cdot\vec{\nabla})\psi + m\bar{\psi}\psi \right]$$

Recalling that $\vec{\alpha} \equiv \gamma^0\vec{\gamma}$ and $\beta \equiv \gamma^0$,

$$H = \int d^3x \left[-i\psi^\dagger \gamma^0 \vec{\gamma}\cdot\vec{\nabla}\psi + m\psi^\dagger \gamma^0 \psi \right]$$

$$H = \int d^3x \, \psi^\dagger \left[-i\vec{\alpha}\cdot\vec{\nabla} + \beta m \right] \psi$$

The term inside the brackets is the Hamiltonian of the Dirac Equation in it's single-particle wave formulation. Call it h_D. The the wave-mechanical version of the Dirac Equation can be written in eigenvalue form as

$$h_D \psi = i\hbar \frac{\partial}{\partial t} \psi$$

and we can identify $i\hbar \frac{\partial}{\partial t}$ as the energy operator. Hence,

$$H = \int d^3x \, \omega_p \, \psi^\dagger \psi$$

$$H = \int d^3x \, \omega_p \left[\int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{p'}}} \sum_{s'} \left(a_{p'}^{s'\dagger} u^{s'\dagger}(p') e^{ip'\cdot x} + b_{p'}^{s'} v^{s'\dagger}(p') e^{-ip'\cdot x} \right) \right.$$
$$\left. \cdot \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \sum_{s} \left(a_p^{s} u^{s}(p) e^{-ip\cdot x} + b_p^{s\dagger} v^{s}(p) e^{ip\cdot x} \right) \right]$$

$$H = \int d^3x \, \omega_p \left[\int\int \frac{d^3p' \, d^3p}{(2\pi)^6 2\sqrt{\omega_p \omega_{p'}}} \sum_{s,s'} \left(a_{p'}^{s'\dagger} a_p^{s} u^{s'\dagger}(p') u^{s}(p) e^{i(p'-p)\cdot x} \right.\right.$$
$$+ a_{p'}^{s'\dagger} b_p^{s\dagger} u^{s'\dagger}(p') v^{s}(p) e^{i(p'+p)\cdot x}$$
$$+ b_{p'}^{s'} a_p^{s} v^{s'\dagger}(p') u^{s}(p) e^{-i(p'+p)\cdot x}$$
$$\left.\left. + b_{p'}^{s'} b_p^{s\dagger} v^{s'\dagger}(p') v^{s}(p) e^{-i(p'-p)\cdot x} \right) \right]$$

By direct computation that I won't work out here $u^{s'\dagger}(p') v^{s}(p) = v^{s'\dagger}(p') u^{s}(p) = 0$, so the middle two terms vanish.

$$H = \int d^3x\, \omega_p \left[\iint \frac{d^3p'\, d^3p}{(2\pi)^6 2\sqrt{\omega_p \omega_{p'}}} \sum_{s',s} \left(a^{s'\dagger}_{p'} a^s_p u^{s'\dagger}(p') u^s(p) e^{i(p'-p)\cdot x} + b^{s'}_{p'} b^{s\dagger}_p v^{s'\dagger}(p') v^s(p) e^{-i(p'-p)\cdot x} \right) \right]$$

Integrate over $\vec{x}$.

$$H = \omega_p \iint \frac{d^3p'\, d^3p}{(2\pi)^6 2\sqrt{\omega_{p'}\omega_p}} \sum_{s',s} \left(a^{s'\dagger}_{p'} a^s_p u^{s'\dagger}(p') u^s(p) (2\pi)^3 \delta^3(\vec{p}' - \vec{p}) + b^{s'}_{p'} b^{s\dagger}_p v^{s'\dagger}(p') v^s(p) (2\pi)^3 \delta^3(\vec{p}' - \vec{p}) \right)$$

Integrate over $\vec{p}'$.

$$H = \omega_p \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_p} \sum_{s',s} \left(a^{s'\dagger}_p a^s_p u^{s'\dagger}(p) u^s(p) + b^{s'}_p b^{s\dagger}_p v^{s'\dagger}(p) v^s(p) \right)$$

Finally, use $u^{s'\dagger}(p) u^s(p) = v^{s'\dagger}(p) v^s(p) = 2\omega_p \delta_{s's}$.

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a^{s\dagger}_p a^s_p - b^s_p b^{s\dagger}_p \right) \omega_p$$

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a^{s\dagger}_p a^s_p + b^{s\dagger}_p b^s_p \right)}$$

In the last line I used the anticommutator relationship to switch b and $b^\dagger$ where $p' = p$ and $s' = s$ has already been established.