The Dirac Propagator

We wish to know the probability amplitude for the scenario of a Dirac field propagating between the spacetime locations x^μ and y^μ. Recall that in Dirac Theory it is $\psi\bar{\psi}$, not $\psi\psi^\dagger$, that is the Lorentz Invariant quantity.

To find the propagator, we first find a Green's function of the Dirac operator, then do a Fourier transform.

Let's begin the calculation with $\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle$ and $\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle$.

From p. 55-56 of Peskin-Schroeder:

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_s u_a^s(p)\bar{u}_b^s(p)\,e^{-ip\cdot(x-y)}$$

Use the spin-sum calculation from p. 49 of Peskin-Schroeder.

$$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle = (i\partial\!\!\!/_x + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\,e^{-ip\cdot(x-y)}$$

Similarly,

$$\langle 0|\bar{\psi}_b(y)\psi_a(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_s v_a^s(p)\bar{v}_b^s(p)\,e^{ip\cdot(x-y)}$$

$$= -(i\partial\!\!\!/_x + m)_{ab}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\,e^{ip\cdot(x-y)}$$

These two terms together cover both the possibilities of propagation from x to y and also from y to x. Later, we'll see how causality is built in to allow only one or the other depending on if $x^0 > y^0$ or $x^0 < y^0$. They may be written more compactly as

$$S_R(x-y) \equiv \Theta(x^0-y^0)\,\langle 0|\{\psi_a(x), \bar{\psi}_b(y)\}|0\rangle$$

$$= \Theta(x^0-y^0)\,\langle 0|\left[\psi_a(x)\bar{\psi}_b(y) + \bar{\psi}_b(y)\psi_a(x)\right]|0\rangle$$

$$= \Theta(x^0-y^0)\left[\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle + \langle 0|\bar{\psi}_b(y)\psi(x)|0\rangle\right]$$

$$= \Theta(x^0-y^0)\left[(i\partial\!\!\!/_x+m)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip\cdot(x-y)} - (i\partial\!\!\!/_x+m)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{ip\cdot(x-y)}\right]$$

$$= \Theta(x^0-y^0)(i\partial\!\!\!/_x+m)\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right]$$

This is so similar to the Klein-Gordon Propagator calculation that I'll follow with Cauchy's Integral formula and contour integration in the same manner.

Cauchy's Integral formula is

$$\oint \frac{f(z)\,dz}{z-a} = 2\pi i\, f(a)$$

So let $z = p$, $f(z) = e^{ip\cdot(x-y)}$, and $a = \pm E_p$.

$$S_R(x-y) = \Theta(x^0-y^0)(i\partial\!\!\!/_x + m)\int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2E_p} e^{-ip(x-y)}\Big|_{p^0=E_p} + \frac{1}{-2E_p} e^{-ip\cdot(x-y)}\Big|_{p^0=-E_p}\right]$$

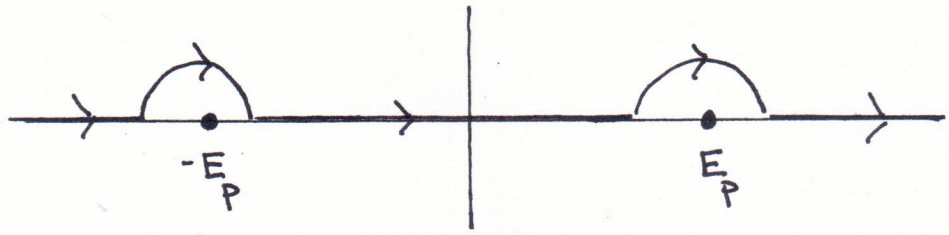
$$= \Theta(x^0-y^0)(i\partial\!\!\!/_x + m)\int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2E_p}\oint \frac{dp^0}{2\pi i}\frac{e^{-ip\cdot(x-y)}}{p-E_p} - \frac{1}{2E_p}\oint \frac{dp^0}{2\pi i}\frac{e^{-ip\cdot(x-y)}}{p-(-E_p)}\right]$$

$$= \Theta(x^0-y^0)(i\partial\!\!\!/_x + m)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\oint \frac{dp^0}{2\pi i} e^{-ip\cdot(x-y)}\left[\frac{1}{p-E_p} - \frac{1}{p+E_p}\right]$$

$$= \Theta(x^0-y^0)(i\partial\!\!\!/_x + m)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\oint \frac{dp^0}{2\pi i} e^{-ip\cdot(x-y)}\left[\frac{-2E_p}{p^2-m^2}\right]$$

$$= \Theta(x^0-y^0)(i\partial\!\!\!/_x + m)\int \frac{d^3p}{(2\pi)^3}\oint \frac{dp^0}{2\pi i} e^{-ip\cdot(x-y)}\frac{-1}{p^2-m^2}$$

The p^0-integral here is performed over



For $x^0 > y^0$, the contour closes below and contains the poles. For $x^0 < y^0$, the contour closes above and the integral is zero.

$$S_R(x-y) = (i\not{\partial}_x + m)\int \frac{d^3p}{(2\pi)^3} \oint \frac{dp^0}{2\pi i}\, e^{-ip\cdot(x-y)} \frac{-1}{p^2-m^2}$$

$$= (i\not{\partial}_x + m)\int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)} \frac{i}{p^2-m^2}$$

$$= (i\not{\partial}_x + m)\, D_R(x-y)$$

That just got rid of the step-function.

The Dirac operator is $(i\not{\partial}_x - m)$ and the following illustrates how $S_R(x-y)$ is a Green's function of the Dirac operator:

$$(i\not{\partial}_x - m)S_R(x-y) = (i\not{\partial}_x - m)(i\not{\partial}_x + m)D_R(x-y)$$

$$= (-\not{\partial}_x\not{\partial}_x - m^2)\, D_R(x-y) \qquad (\not{\partial}_x\not{\partial}_x = \partial_x^2)$$

$$= -(\partial^2 + m^2)\, D_R(x-y)$$

$$= -(-i\delta^{(4)}(x-y))$$

$$= i\delta^{(4)}(x-y).$$

where I used the fact that the same calculation was performed for the K.G. Propagator.

Using this, Fourier expand $S_R(x-y)$.

$$S_R(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \tilde{S}_R(p)$$

Apply the Dirac operator

$$(i\partial\!\!\!/_x - m) S_R(x-y) = \int \frac{d^4p}{(2\pi)^4} (i\partial\!\!\!/_x - m) e^{-ip\cdot(x-y)} \tilde{S}_R(p)$$

Compare this to the Green's function result we just saw

$$\begin{aligned} i\delta^{(4)}(x-y) &= (i\partial\!\!\!/_x - m) S_R(x-y) \\ &= (i\partial\!\!\!/_x - m)(i\partial\!\!\!/_x + m) D_R(x-y) \\ &= (p^2 - m^2) \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i}{p^2 - m^2} \\ &= \int \frac{d^4p}{(2\pi)^4} i e^{ip\cdot(x-y)} \end{aligned}$$

Hence, $(p\!\!\!/ - m) \tilde{S}_R(p) = i$

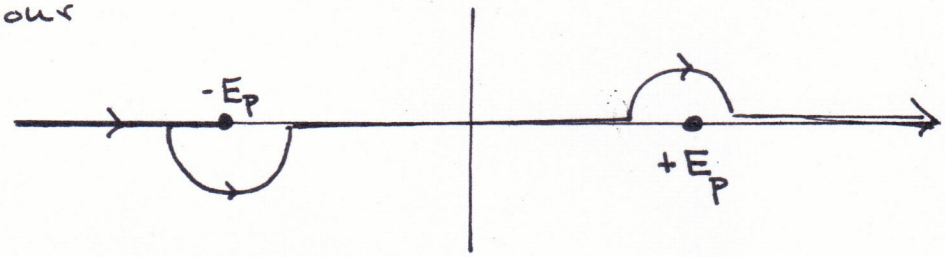
$$\tilde{S}_R(p) = \frac{i}{p\!\!\!/ - m} \cdot \frac{p\!\!\!/ + m}{p\!\!\!/ + m}$$

$$\tilde{S}_R(p) = \frac{i(p\!\!\!/ + m)}{p^2 - m^2}$$

Now that the Fourier Transform of $S_R(x-y)$ is known, put it back in

$$S_R(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \frac{i(\not{p}+m)}{p^2-m^2}$$

where the p^0 integral is now along the contour



$$\Rightarrow S_R(x-y) = \begin{cases} \langle 0|\Psi(x)\bar{\Psi}(y)|0\rangle & \text{for } x^0 > y^0 \text{ (close contour below)} \\ -\langle 0|\bar{\Psi}(y)\Psi(x)|0\rangle & \text{for } x^0 < y^0 \text{ (close contour above)} \end{cases}$$

$$\Rightarrow \boxed{S_R(x-y) \equiv \langle 0|T[\Psi(x)\bar{\Psi}(y)]|0\rangle}$$