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Physics 831
HW 6

1(i) <u>Show $\gamma^\mu \gamma_\mu = 4$.</u>

This can be shown directly from the anticommutator relationship $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\overleftrightarrow{1}$.

$$\Rightarrow \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\overleftrightarrow{1}$$

$$\Rightarrow g_{\rho\nu}\left[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\overleftrightarrow{1}\right]$$

$$\Rightarrow \gamma^\mu\gamma_\rho + \gamma_\rho\gamma^\mu = 2g^\mu_\rho\overleftrightarrow{1}$$

Now consider the particular case where $\mu = \rho$.

$$\Rightarrow \gamma^\mu\gamma_\mu = g^\mu_\mu\overleftrightarrow{1}$$

$$= \gamma^0\gamma_0 - \gamma^1\gamma_1 - \gamma^2\gamma_2 - \gamma^3\gamma_3$$

$$= g^0_0\overleftrightarrow{1} - g^1_1\overleftrightarrow{1} - g^2_2\overleftrightarrow{1} - g^3_3\overleftrightarrow{1}$$

$$= \overleftrightarrow{1} + \overleftrightarrow{1} + \overleftrightarrow{1} + \overleftrightarrow{1}$$

$$\boxed{\gamma^\mu\gamma_\mu = 4\cdot\overleftrightarrow{1}}$$

(ii) <u>Show $\gamma^\mu \not{a} \gamma_\mu = -2\not{a}$</u>

$$\gamma^\mu \not{a} \gamma_\mu = \gamma^\mu a^\nu \gamma_\nu \gamma_\mu$$

$$= \gamma^\mu a^\nu (2 g_{\mu\nu} - \gamma_\mu \gamma_\nu)$$

$$= 2\gamma_\nu a^\nu - \gamma^\mu a^\nu \gamma_\mu \gamma_\nu$$

Notice that a^ν is just a scalar component of a 4-vector, so it commutes with γ-matrices.

$$= 2\gamma_\nu a^\nu - \gamma^\mu \gamma_\mu a^\nu \gamma_\nu$$

$$= 2\not{a} - 4\not{a}$$

$$\boxed{\gamma^\mu \not{a} \gamma_\mu = -2\not{a}}$$

(iii) <u>Show $\gamma^\mu \not{a} \not{b} \gamma_\mu = 4a \cdot b$.</u>

$$\gamma^\mu \not{a} \not{b} \gamma_\mu = \gamma^\mu \not{a} b^\sigma \gamma_\sigma \gamma_\mu$$

$$= \gamma^\mu \not{a} b^\sigma [2 g_{\sigma\mu} - \gamma_\mu \gamma_\sigma]$$

$$= 2\gamma^\mu \not{a} b^\sigma g_{\sigma\mu} - \gamma^\mu \not{a} b^\sigma \gamma_\mu \gamma_\sigma$$

$$= 2\gamma^\mu \not{a} b_\mu - \gamma^\mu \not{a} \gamma_\mu \not{b}$$

$$= 2\not{b}\not{a} + 2\not{a}\not{b}$$

$$= 2b^\mu \gamma_\mu a^\rho \gamma_\rho + 2\not{a}\not{b}$$

$$= 2b^\mu a^\rho [2g_{\mu\rho} - \gamma_\rho \gamma_\mu] + 2\not{a}\not{b}$$

$$= 4b^\mu a^\rho g_{\mu\rho} - 2b^\mu a^\rho \gamma_\rho \gamma_\mu + 2\not{a}\not{b}$$

$$= 4b^\mu a_\mu - 2\not{a}\not{b} + 2\not{a}\not{b}$$

$$\boxed{\gamma^\mu \not{a}\not{b}\gamma_\mu = 4a\cdot b}$$

(iv) <u>Show $\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}$</u>

$$\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = \gamma^\mu \not{a}\not{b} c^\sigma \gamma_\sigma \gamma_\mu$$

$$= \gamma^\mu \not{a}\not{b} c^\sigma [2g_{\sigma\mu} - \gamma_\mu \gamma_\sigma]$$

$$= \gamma^\mu \not{a}\not{b} c^\sigma 2g_{\sigma\mu} - \gamma^\mu \not{a}\not{b} c^\sigma \gamma_\mu \gamma_\sigma$$

$$= 2\gamma^\mu \not{a}\not{b} c_\mu - \gamma^\mu \not{a}\not{b}\gamma_\mu \not{c}$$

$$= 2\not{c}\not{a}\not{b} - 4a\cdot b\not{c}$$

$$= 2\not{c} a^\rho \gamma_\rho b^\nu \gamma_\nu - 4a\cdot b\not{c}$$

$$= 2\not{c} a^\rho b^\nu \gamma_\rho \gamma_\nu - 4a\cdot b\not{c}$$

$$= 2\not{c}\, a^\rho b^\nu \left[2 g_{\rho\nu} - \gamma_\nu \gamma_\rho \right] - 4 a\cdot b\, \not{c}$$

$$= 4\not{c}\, a^\rho b^\nu g_{\rho\nu} - 2\not{c}\, a^\rho b^\nu \gamma_\nu \gamma_\rho - 4 a\cdot b\, \not{c}$$

$$= 4\not{c}\, a\cdot b - 2\not{c}\not{b}\not{a} - 4 a\cdot b\, \not{c}$$

$$\boxed{\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}}$$

Tools and formulae for loop calculations

I. FEYNMAN PARAMETERIZATION

Feynman parameterization 1:

To rewrite a product denominator into a symmetric form of the integration momentum, the following Feynman parameteerization formulae are useful:

$$\frac{1}{A_1 A_2 ... A_n} = \Gamma(n) \int_0^1 dx_1 dx_2 ... dx_n \, \frac{\delta(1-\sum_{i=1}^n x_i)}{(\sum_{i=1}^n A_i x_i)^n} \, ,$$

$$\frac{1}{A_1^{m_1} A_2^{m_2} ... A_n^{m_n}} = \frac{\Gamma(\sum_{j=1}^n m_j)}{\Gamma(m_1)\Gamma(m_2)...\Gamma(m_n)} \int_0^1 dx_1 dx_2 ... dx_n \, \frac{\delta(1-\sum_{i=1}^n x_i) \, \Pi_j x_j^{m_j-1}}{(\sum_{i=1}^n A_i x_i)^{\sum_{j=1}^n m_j}} \, .$$

Simplest examples:

$$\frac{1}{A_1 A_2} = \int_0^1 dx_1 dx_2 \, \frac{\delta(1-x_1-x_2)}{(A_1 x_1 + A_2 x_2)^2} = \int_0^1 \frac{dx_1}{(A_1 x_1 + A_2(1-x_1))^2} \, ,$$

$$\frac{1}{A_1^{m_1} A_2^{m_2}} = \frac{\Gamma(m_1+m_2)}{\Gamma(m_1)\Gamma(m_2)} \int_0^1 dx_1 dx_2 \, \frac{\delta(1-x_1-x_2) \, x_1^{m_1-1} x_2^{m_2-1}}{(A_1 x_1 + A_2 x_2)^{m_1+m_2}} \, .$$

Feynman parameterization 2:

$$\frac{i}{p^2 - m^2 + i\epsilon} = \int_0^\infty d\alpha \, e^{i\alpha(p^2-m^2+i\epsilon)}. \tag{1}$$

II. FOUR-DIMENSIONAL INTEGRALS

(1) Wick rotation

$$\ell^0 = i\ell_E^0, \; \vec{\ell} = \vec{\ell}_E, \quad d^4\ell = i d^4\ell_E = i \, d\Omega_3 \, \ell_E^3 d\ell_E = i \, 2\pi^2 \, \ell_E^3 d\ell_E,$$

(2). 4D convergent integrals:

$$\int \frac{d^4\ell}{(2\pi)^4} \, \frac{\ell^\mu}{[\ell^2-\Delta]^m} f(\ell^2) = 0, \quad \int \frac{d^4\ell}{(2\pi)^4} \, \frac{\ell^\mu \ell^\nu}{[\ell^2-\Delta]^m} f(\ell^2) = \frac{1}{4} g^{\mu\nu} \int \frac{d^4\ell}{(2\pi)^4} \, \frac{\ell^2}{[\ell^2-\Delta]^m} f(\ell^2).$$

$$\int \frac{d^4\ell}{(2\pi)^4} \, \frac{1}{[\ell^2-\Delta]^m} = \frac{i(-1)^m}{(4\pi)^2} \, \frac{1}{(m-1)(m-2)} \, \frac{1}{\Delta^{m-2}} \quad (m>2).$$

$$\int \frac{d^4\ell}{(2\pi)^4} \, \frac{\ell^2}{[\ell^2-\Delta]^m} = \frac{i(-1)^{m-1}}{(4\pi)^2} \, \frac{1}{(m-1)(m-2)(m-3)} \, \frac{1}{\Delta^{m-3}} \quad (m>3).$$

These results will be generalized to n-dimensions.

III. DIMENSIONAL REGULARIZATION

In a n-dimensional Minkowski space,

$$g_{\mu\nu}g^{\mu\nu} = n = 4 - 2\epsilon.$$

A. Dirac algebra

The n-dim Dirac algebra is defined as

$$\begin{aligned}
&\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\ \mathrm{I},\\
&\gamma^\mu\gamma_\mu = g^\mu_\mu \mathrm{I} = n\mathrm{I}, \quad \gamma^\mu \not{a}\gamma_\mu = (2-n)\not{a},\\
&\gamma^\mu \not{a}\not{b}\gamma_\mu = 4a\cdot b\mathrm{I} + (n-4)\not{a}\not{b},\\
&\gamma^\mu \not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a} - (n-4)\not{a}\not{b}\not{c}.
\end{aligned}$$

The traces then remain the same form as in 4D by convention

$$\begin{aligned}
&\mathrm{Tr}\mathrm{I} = 4, \quad \mathrm{Tr}\gamma_\mu\gamma_\nu = 4g_{\mu\nu},\\
&\mathrm{Tr}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma = 4(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}).
\end{aligned}$$

B. n-dimensional Euclidean space

Angular variables:

$$\begin{aligned}
&d^n\ell_E = \ell_E^{n-1}d\ell_E\ d\Omega_n\\
&d\Omega_n = (\sin^{n-2}\theta_{n-1}\ \sin^{n-3}\theta_{n-2}\ \ldots \sin\theta_2)\ (d\theta_{n-1}d\theta_{n-2}\ldots d\theta_2 d\theta_1)\\
&0 \le \theta_1 \le 2\pi, \quad 0 \le \theta_{2,3,\ldots,n-1} \le \pi.
\end{aligned}$$

And the angular integration

$$\int_0^\pi \sin^m\theta\ d\theta = \sqrt{\pi}\ \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})}, \quad \int d\Omega_n = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int_0^\pi \sin^{n-2}\theta_{n-1}d\theta_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})},$$

$$\int d\Omega_3 = \frac{(4\pi)^{1-\epsilon}}{\Gamma(1-\epsilon)}\int_0^1 v^{-\epsilon}(1-v)^{-\epsilon}dv, \quad v = \frac{1}{2}(1+\cos\theta_2).$$

C. Loop integrals

n-dimensional loop integrals in Minkowski space read

$$\int \frac{d^n\ell}{(2\pi)^n} \frac{(\ell^2)^r}{[\ell^2 - \Delta + i\epsilon]^m} = \frac{i(-1)^{m-r}}{(4\pi)^{n/2}} \frac{\Gamma(r+n/2)}{\Gamma(n/2)} \frac{\Gamma(m-r-n/2)}{\Gamma(m)} \frac{1}{(\Delta - i\epsilon)^{m-r-n/2}}.$$

This reproduces the early results for $n = 4$. Note that the effect of the Wick rotation to Euclidean space simply gives a factor $i(-1)^{m-r}$.

D. Gamma function

(1) <u>Gamma function</u>

Definition:

$$\Gamma(x) = \int_0^1 e^{-y}\, y^{x-1}\, dy, \quad x > 0.$$

Properties:

$$\Gamma(n+1) = n!\ (n = \text{integer}), \quad \Gamma(x+1) = x\Gamma(x),$$
$$\Gamma(1) = 1, \quad \Gamma(1/2) = \sqrt{\pi},$$
$$\Gamma(1+\epsilon) = 1 - \gamma_E\epsilon + \frac{1}{2}\,(\gamma_E^2 + \frac{\pi^2}{6}) + \ldots$$

where the Euler's constant $\gamma_E \approx 0.5772...$

(2) <u>Beta function</u>

Definition:

$$B(r,m) = \int_0^1 x^{r-1}\,(1-x)^{m-1}\, dx = \frac{\Gamma(r)\Gamma(m)}{\Gamma(r+m)}.$$