2. Gordon Decomposition

Proposition: Let $u_1(p_1)$ and $u_2(p_2)$ be two Dirac Spinors. Then

$$\bar{u}_1 \gamma^\mu u_2 = \frac{1}{2m} \bar{u}_1 \left(P^\mu + i q_\nu \sigma^{\mu\nu}\right) u_2$$

where $P^\mu \equiv p_1^\mu + p_2^\mu$, $q^\mu = p_1^\mu - p_2^\mu$, and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

Proof: I will begin with the 2nd term on the right-hand-side of the decomposition; the result will follow.

$$\frac{1}{2m} \bar{u}_1 i q_\nu \sigma^{\mu\nu} u_2$$

$$= \bar{u}_1 \frac{i(p_{1\nu} - p_{2\nu}) \frac{i}{2}[\gamma^\mu, \gamma^\nu]}{2m} u_2$$

$$= \bar{u}_1 \frac{(p_{2\nu} - p_{1\nu})(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)}{4m} u_2$$

$$= \bar{u}_1 \frac{p_{2\nu}\gamma^\mu\gamma^\nu - p_{2\nu}\gamma^\nu\gamma^\mu - p_{1\nu}\gamma^\mu\gamma^\nu + p_{1\nu}\gamma^\nu\gamma^\mu}{4m} u_2$$

Use the Dirac equation and its conjugate.

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \Rightarrow (-\not{p} - m)\psi = 0$$

$$\bar{\psi}(-i\partial_\mu \gamma^\mu - m) = 0 \Rightarrow \bar{\psi}(\not{p} - m) = 0$$

$$= \bar{u}_1 \frac{\gamma^\mu \not{p}_2 - \not{p}_2 \gamma^\mu - \gamma^\mu \not{p}_1 + \not{p}_1 \gamma^\mu}{4m} u_2$$

$$= \bar{u}_1 \frac{(-\not{p}_2 \gamma^\mu - \gamma^\mu \not{p}_1)}{4m} u_2 + \bar{u}_1 \frac{(\gamma^\mu \not{p}_2 + \not{p}_1 \gamma^\mu)}{4m} u_2$$

$$= \bar{u}_1 \frac{(-\not{p}_2 \gamma^\mu - \gamma^\mu \not{p}_1)}{4m} u_2 + \bar{u}_1 \frac{(2m\gamma^\mu)}{4m} u_2$$

$$= \underbrace{-\frac{\bar{u}_1 \not{p}_2 \gamma^\mu u_2}{4m}}_{(\mathrm{I})} \underbrace{- \frac{\bar{u}_1 \gamma^\mu \not{p}_1 u_2}{4m}}_{(\mathrm{II})} + \frac{\bar{u}_1 \gamma^\mu u_2}{2}$$

I'll now deal with (I) and (II) separately, and put everything back together at the end.

(I):

$$\bar{u}_1 \frac{\not{p}_2 \gamma^\mu u_2}{4m} = \bar{u}_1 \frac{p_{2\nu} \gamma^\nu \gamma^\mu u_2}{4m}$$

$$= \frac{\bar{u}_1 p_{2\nu} [2g^{\nu\mu} - \gamma^\mu \gamma^\nu] u_2}{4m}$$

$$= \frac{\bar{u}_1 2 p_{2\nu} g^{\nu\mu} u_2}{4m} - \frac{\bar{u}_1 p_{2\nu} \gamma^\mu \gamma^\nu u_2}{4m}$$

$$= \frac{u_1 p_2^\mu u_2}{2m} - \frac{\bar{u}_1 \gamma^\mu \not{p}_2 u_2}{4m}$$

$$= \frac{\bar{u}_1 p_2^\mu u_2}{2m} + \frac{\bar{u}_1 \gamma^\mu m u_2}{4m}$$

$$= \frac{\bar{u}_1 p_2^\mu u_2}{2m} + \frac{\bar{u}_1 \gamma^\mu u_2}{4}$$

(II)

$$\frac{\bar{u}_1 \gamma^\mu \not{p}_1 u_2}{4m} = \frac{\bar{u}_1 \gamma^\mu p_{1\nu} \gamma^\nu u_2}{4m}$$

$$= \bar{u}_1 \frac{p_{1\nu} \gamma^\mu \gamma^\nu}{4m} u_2$$

$$= \bar{u}_1 \frac{p_{1\nu} \left[2g^{\mu\nu} - \gamma^\nu \gamma^\mu \right]}{4m} u_2$$

$$= \frac{\bar{u}_1 2 p_1^\mu u_2}{4m} - \frac{\bar{u}_1 p_{1\nu} \gamma^\nu \gamma^\mu u_2}{4m}$$

$$= \frac{\bar{u}_1 p_1^\mu u_2}{2m} - \frac{\bar{u}_1 \not{p}_1 \gamma^\mu u_2}{4m}$$

$$= \frac{\bar{u}_1 p_1^\mu u_2}{2m} - \frac{\bar{u}_1 \gamma^\mu u_2}{4}$$

Put (I) and (II) back into the original equation.

$$\frac{1}{2m}\bar{u}_1\left(iq_\nu\sigma^{\mu\nu}\right)u_2 = -\left[\frac{\bar{u}_1 p_2^\mu u_2}{2m} + \cancel{\frac{\bar{u}_1\gamma^\mu u_2}{4}} + \frac{\bar{u}_1 p_1^\mu u_2}{2m} - \cancel{\frac{\bar{u}_1\gamma^\mu u_2}{4}}\right] + \frac{\bar{u}_1\gamma^\mu u_2}{2}$$

$$= -\frac{\bar{u}_1 P^\mu u_2}{2m} + \frac{\bar{u}_1\gamma^\mu u_2}{2}$$

$$\Rightarrow \boxed{\frac{1}{2m}\bar{u}_1\left(iq_\nu\sigma^{\mu\nu} + P^\mu\right)u_2 = \frac{\bar{u}_1\gamma^\mu u_2}{2}}$$