Parity: $(t, \vec{x}) \to (t, -\vec{x})$

How does a parity transformation $P:(t,\vec{x})\to(t,-\vec{x})$ act on a Dirac Field? P reverses the momentum of a particle without changing its spin, thus changing handedness. As such, P transforms a state from

$$a^{s\dagger}_{\vec{p}}|0\rangle$$

to $$a^{s\dagger}_{-\vec{p}}|0\rangle.$$

So whatever P is, it transforms a creation/annihilation operator as

$$P a^s_p P = \eta_a a^s_{-p}$$

where η_a is just a phase factor that must satisfy $|\eta_a|^2 = \pm 1$, like usual for phase factors.

To act on a Dirac field, P must be a 4×4 matrix. To find P and the value of the phase factor, let's see what P does to $\psi(x)$ based on what we know so far.

$$P\psi(x)P = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(\eta_a a^s_{-p}u^s(p)e^{-ip\cdot x} + \eta^*_b b^{s\dagger}_{-p}v^s(p)e^{ip\cdot x}\right)$$

$$= \int \frac{d^3\tilde{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(\eta_a a^s_{\tilde{p}}\gamma^0 u^s(\tilde{p})e^{-i\tilde{p}\cdot(t,\vec{x})} - \eta^*_b b^{s\dagger}_{\tilde{p}}\gamma^0 v^s(\tilde{p})e^{i\tilde{p}\cdot(t,-\vec{x})}\right)$$

To have the form of a parity transformation, this last line should be some constant matrix times $\psi(t,-\vec{x})$.

$$\Rightarrow \boxed{P\psi(t,\vec{x})P = \eta_a \gamma^0 \psi(t,-\vec{x})}$$

Similarly,

$$\Rightarrow \boxed{P\bar{\psi}(t,\vec{x})P = \eta_a^* \bar{\psi}(t,-\vec{x})\gamma^0}$$

Scalar Bilinear:

$$P\bar{\psi}(t,\vec{x})\psi(t,\vec{x})P = P\Big[\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\sum_s \left(a_p^s u^s(p) e^{-ip\cdot x} + b_p^{s\dagger} v^s(p) e^{ip\cdot x}\right) \int \frac{d^3p'}{(2\pi)^3}\frac{1}{2E_{p'}}\sum_{s'}\left(a_{p'}^{s'\dagger} u^{s'}(p') e^{-ip'\cdot x} + b_{p'}^{s'} v^{s'}(p') e^{ip'\cdot x}\right)\Big] P$$

$$= \left(\eta_a^* \bar{\psi}(t,-\vec{x})\gamma^0\right)\left(\eta_a \gamma^0 \psi(t,-\vec{x})\right)$$

$$= |\eta_a|^2 \bar{\psi}(t,-\vec{x}) \underbrace{\gamma^0\gamma^0}_{\cdot 1} \psi(t,-\vec{x})$$

$$= \psi(t,-\vec{x})\psi(t,-\vec{x})$$

Vector Bilinear $(\bar{\psi}\gamma^\mu\psi)$

$$P\bar{\psi}(t,\vec{x})\gamma^\mu\psi(t,\vec{x})P = \eta_a^* \bar{\psi}(t,-\vec{x})\gamma^0\gamma^\mu\eta_a\gamma^0\psi(t,-\vec{x})$$

$$= \bar{\psi}(t,-\vec{x})\gamma^0\gamma^\mu\gamma^0\psi(t,-\vec{x})$$

The problem now reduces to dealing with $\gamma^0\gamma^\mu\gamma^0$.

In the Weyl Representation,

$$\gamma^0 = \begin{bmatrix} 0 & \overrightarrow{1} \\ \overleftarrow{1} & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}$$

$$\Rightarrow \gamma^0\gamma^0\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \gamma^0$$

and

$$\Rightarrow \gamma^0\gamma^i\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{bmatrix} = -\gamma^i$$

Thus,

$$P\bar{\psi}(t,\vec{x})\gamma^\mu\psi(t,\vec{x})P = \begin{cases} +\bar{\psi}(t,-\vec{x})\gamma^\mu\psi(t,-\vec{x}) & \text{for } \mu = 0 \\ -\bar{\psi}(t,-\vec{x})\gamma^\mu\psi(t,-\vec{x}) & \text{for } \mu = 1,2,3 \end{cases}$$

Tensor Bilinear ($\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$)

$$P\bar{\Psi}\sigma^{\mu\nu}\Psi P = P\bar{\Psi}\,\frac{i}{2}[\gamma^\mu,\gamma^\nu]\Psi P$$

$$= \frac{i}{2} P\bar{\Psi}\gamma^\mu\gamma^\nu\Psi P - \frac{i}{2} P\bar{\Psi}\gamma^\nu\gamma^\mu\Psi P$$

$$= \frac{i}{2}\bar{\Psi}(t,-\vec{x})\gamma^0\gamma^\mu\gamma^\nu\gamma^0\Psi(t,-\vec{x}) - \frac{i}{2}\bar{\Psi}(t,-\vec{x})\gamma^0\gamma^\nu\gamma^\mu\gamma^0\Psi(t,-\vec{x})$$

Now deal with the γ-matrices.

$$\gamma^0\gamma^\mu\gamma^\nu\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^\nu \\ -\sigma^\nu & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

If $\mu=0$ and $\nu=0$,

$$\gamma^0\gamma^0\gamma^0\gamma^0 = \gamma^0\gamma^0.$$

If $\mu=0$ and $\nu=i$,

$$\gamma^0\gamma^0\gamma^i\gamma^0 = \gamma^i\gamma^0 = -\gamma^0\gamma^i.$$

If $\nu=0$ and $\mu=i$,

$$\gamma^0\gamma^i\gamma^0\gamma^0 = \gamma^0\gamma^i = -\gamma^i\gamma^0.$$

If $\mu=i$ and $\nu=j$,

$$\gamma^0\gamma^i\gamma^j\gamma^0 = (\gamma^0\gamma^i\gamma^0)(\gamma^0\gamma^j\gamma^0)$$

$$= (-\gamma^i)(-\gamma^j)$$

$$= \gamma^i\gamma^j$$

All together,

$$P\bar{\Psi}\sigma^{\mu\nu}\Psi P = (-1)^\mu(-1)^\nu\,\bar{\Psi}(t,-\vec{x})\sigma^{\mu\nu}\Psi(t,-\vec{x})$$

for $(-1)^\mu = \begin{cases} 1 & \text{for } \mu=0 \\ -1 & \text{for } \mu=1,2,3 \end{cases}$

Psendoscalar Bilinear ($i\bar{\psi}\gamma^5\psi$)

$$iP\bar{\psi}\gamma^5\psi P = i\bar{\psi}(t,-\vec{x})\gamma^0\gamma^5\gamma^0\psi(t,-\vec{x})$$

The problem reduces to solving $\gamma^0\gamma^5\gamma^0$. In the Weyl representation

$$\gamma^5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Hence,

$$\gamma^0\gamma^5\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -\gamma^5$$

and

$$\boxed{iP\bar{\psi}\gamma^5\psi P = -i\bar{\psi}(t,-\vec{x})\gamma^5\psi(t,-\vec{x})}$$

Psendovector Bilinear ($\bar{\psi}\gamma^\mu\gamma^5\psi$)

$$P\bar{\psi}\gamma^\mu\gamma^5\psi = \bar{\psi}\gamma^0\gamma^\mu\gamma^5\gamma^0\psi$$

$$\gamma^0\gamma^\mu\gamma^5\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\sigma^\mu & 0 \\ 0 & \sigma^\mu \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\sigma^\mu \\ \sigma^\mu & 0 \end{bmatrix}$$

$$= -\gamma^\mu$$

$$\Rightarrow \boxed{P\bar{\psi}\gamma^\mu\gamma^5\psi P = -\bar{\psi}(t,-\vec{x})\gamma^\mu\psi(t,-\vec{x})}$$

Time Reversal $(t,\vec{x}) \to (-t,\vec{x})$

Dirac fields transform under time-reversal as

$$\begin{cases} T\,\Psi(t,\vec{x})\,T = \gamma^1\gamma^3\,\Psi(-t,\vec{x}) \\ T\,\bar{\Psi}(t,\vec{x})\,T = -\,\bar{\Psi}(-t,\vec{x})\,\gamma^1\gamma^3 \end{cases}$$

Scalar Bilinear $(\bar{\Psi}\Psi)$

$$T\,\bar{\Psi}(t,\vec{x})\,\Psi(t,\vec{x})\,T = \bar{\Psi}(-t,\vec{x})\,(-\gamma^1\gamma^3\gamma^1\gamma^3)\,\Psi(-t,\vec{x})$$

$$\gamma^1\gamma^3\gamma^1\gamma^3 = \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^1\sigma^3\sigma^1\sigma^3 & 0 \\ 0 & \sigma^1\sigma^3\sigma^1\sigma^3 \end{bmatrix}$$

$$\sigma^1\sigma^3\sigma^1\sigma^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \gamma^1\gamma^3\gamma^1\gamma^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \boxed{T\,\bar{\Psi}(t,\vec{x})\,\Psi(t,\vec{x})\,T = \bar{\Psi}(-t,\vec{x})\,\Psi(-t,\vec{x})}$$

Vector Bilinear $(\bar{\Psi}\gamma^\mu\Psi)$

$$T\bar{\Psi}\gamma^\mu\Psi T = -\bar{\Psi}(-t,\vec{x})\gamma^1\gamma^3(\gamma^\mu)^*\gamma^1\gamma^3\Psi(-t,\vec{x})$$

$$\gamma^1\gamma^3(\gamma^\mu)^*\gamma^1\gamma^3 = \begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}\begin{bmatrix} 0 & (\sigma^\mu)^* \\ -(\sigma^\mu)^* & 0 \end{bmatrix}\begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}$$

If $\mu=0$, then

$$\gamma^1\gamma^3(\gamma^0)^*\gamma^1\gamma^3 = \begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\sigma^1\sigma^3 \\ \sigma^1\sigma^3 & 0 \end{bmatrix}\begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \sigma^1\sigma^3\sigma^1\sigma^3 \\ -\sigma^1\sigma^3\sigma^1\sigma^3 & 0 \end{bmatrix}$$

$$\sigma^1\sigma^3\sigma^1\sigma^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$= -\mathbb{1}$$

$$\Rightarrow \gamma^1\gamma^3(\gamma^0)^*\gamma^1\gamma^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\gamma^0$$

If $\mu=1$, then

$$\gamma^1\gamma^3(\gamma^1)^*\gamma^1\gamma^3 = \begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}\begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix}\begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\sigma^1\sigma^3\sigma^1 \\ \sigma^1\sigma^3\sigma^1 & 0 \end{bmatrix}\begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \sigma^1\sigma^3\sigma^1\sigma^1\sigma^3 \\ -\sigma^1\sigma^3\sigma^1\sigma^1\sigma^3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \sigma^1\sigma^3\sigma^3 \\ -\sigma^1\sigma^3\sigma^3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix}$$

$$\Rightarrow \gamma^1\gamma^3(\gamma^1)^*\gamma^1\gamma^3 = \gamma^1$$

If $\mu=2$, then

$$\gamma^1\gamma^3(\gamma^2)^*\gamma^1\gamma^3 = \begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}\begin{bmatrix} 0 & (\sigma^2)^* \\ -(\sigma^2)^* & 0 \end{bmatrix}\begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\sigma^1\sigma^3(\sigma^2)^* \\ \sigma^1\sigma^3(\sigma^2)^* & 0 \end{bmatrix}\begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \sigma^1\sigma^3(\sigma^2)^*\sigma^1\sigma^3 \\ -\sigma^1\sigma^3(\sigma^2)^*\sigma^1\sigma^3 & 0 \end{bmatrix}$$

$$\sigma^1\sigma^3(\sigma^2)^*\sigma^1\sigma^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$= \sigma^2$$

$$\Rightarrow \gamma^1\gamma^3(\gamma^2)^*\gamma^1\gamma^3 = \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} = \gamma^2$$

If $\mu=3$, then

$$\gamma^1\gamma^3(\gamma^3)^*\gamma^1\gamma^3 = \begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}\begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix}\begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -\sigma^1\sigma^3\sigma^3 \\ \sigma^1\sigma^3\sigma^3 & 0 \end{bmatrix}\begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \sigma^1\sigma^3\sigma^3\sigma^1\sigma^3 \\ -\sigma^1\sigma^3\sigma^3\sigma^1\sigma^3 & 0 \end{bmatrix}$$

$$\sigma^1\sigma^3\sigma^3\sigma^1\sigma^3 = \sigma^1\sigma^1\sigma^3 = \sigma^3$$

$$\Rightarrow \gamma^1\gamma^3\gamma^3\gamma^1\gamma^3 = \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix} = \gamma^3$$

Hence, the time-reversal operation affects a vector quantity as

$$T\bar{\psi}\gamma^\mu\psi T = \begin{cases} \bar{\psi}(-t,\vec{x})\gamma^\mu\psi(-t,\vec{x}) & \text{for } \mu=0 \\ -\bar{\psi}(-t,\vec{x})\gamma^\mu\psi(-t,\vec{x}) & \text{for } \mu=1,2,3 \end{cases}$$

Tensor Bilinear $(\bar{\Psi}\sigma^{\mu\nu}\Psi)$

$$T\bar{\Psi}\sigma^{\mu\nu}\Psi T = \frac{i}{2}T\bar{\Psi}\gamma^\mu\gamma^\nu\Psi T - \frac{i}{2}T\bar{\Psi}\gamma^\nu\gamma^\mu\Psi T$$

$$= \frac{i}{2}\left(-\bar{\Psi}(-t,\vec{x})\gamma^1\gamma^3\right)(\gamma^\mu\gamma^\nu)^*\left(\gamma^1\gamma^3\Psi(-t,\vec{x})\right)$$

$$-\frac{i}{2}\left(-\bar{\Psi}(-t,\vec{x})\gamma^1\gamma^3\right)(\gamma^\nu\gamma^\mu)^*\left(\gamma^1\gamma^3\Psi(-t,\vec{x})\right)$$

The problem has reduced to calculating

$$\gamma^1\gamma^3(\gamma^\mu)^*(\gamma^\nu)^*\gamma^1\gamma^3 = \begin{bmatrix}-\sigma^1\sigma^3 & 0\\ 0 & -\sigma^1\sigma^3\end{bmatrix}\begin{bmatrix}0 & (\sigma^\mu)^*\\ (-\sigma^\mu)^* & 0\end{bmatrix}\begin{bmatrix}0 & (\sigma^\nu)^*\\ (-\sigma^\nu)^* & 0\end{bmatrix}\begin{bmatrix}-\sigma^1\sigma^3 & 0\\ 0 & -\sigma^1\sigma^3\end{bmatrix}$$

$$= \begin{bmatrix}0 & -\sigma^1\sigma^3(\sigma^\mu)^*\\ \sigma^1\sigma^3(\sigma^\mu)^* & 0\end{bmatrix}\begin{bmatrix}0 & -(\sigma^\nu)^*\sigma^1\sigma^3\\ (\sigma^\nu)^*\sigma^1\sigma^3 & 0\end{bmatrix}$$

$$= \begin{bmatrix}-\sigma^1\sigma^3(\sigma^\mu)^*(\sigma^\nu)^*\sigma^1\sigma^3 & 0\\ 0 & -\sigma^1\sigma^3(\sigma^\mu)^*(\sigma^\nu)^*\sigma^1\sigma^3\end{bmatrix}$$

Let's look at all 16 elements of $\sigma^1\sigma^3(\sigma^\mu)^*(\sigma^\nu)^*\sigma^1\sigma^3$.

$$\left[\sigma^1\sigma^3(\sigma^\mu)^*(\sigma^\nu)^*\sigma^1\sigma^3\right] = \begin{bmatrix} -1 & \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} & \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} & \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\\ \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} & -1 & \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix} & \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\\ \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} & \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix} & -1 & \begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix}\\ \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} & \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} & \begin{bmatrix}0 & i\\ i & 0\end{bmatrix} & -1\end{bmatrix}$$

Compare this to

$$[\sigma^\mu \bar{\sigma}^\nu] = \begin{vmatrix} 1 & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 1 & \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} & 1 & \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} & 1 \end{vmatrix}$$

And we find

$$\sigma^1\sigma^3(\sigma^\mu)^*(\sigma^\nu)^*\sigma^1\sigma^3 = -(-1)^\mu(-1)^\nu\sigma^\mu\sigma^\nu$$

Substituting back:

$$\gamma^1\gamma^3(\gamma^\mu)^*(\gamma^\nu)^*\gamma^1\gamma^3 = \begin{bmatrix} +(-1)^\mu(-1)^\nu\sigma^\mu\sigma^\nu & 0 \\ 0 & (-1)^\mu(-1)^\nu\sigma^\mu\sigma^\nu \end{bmatrix}$$

And back further:

$$\mathcal{T}\bar{\psi}\sigma^{\mu\nu}\psi\mathcal{T} = \frac{i}{2}\left(\bar{\psi}(-t,\vec{x})\left(-(-1)^\mu(-1)^\nu\sigma^\mu\sigma^\nu\right)\psi(-t,\vec{x})\right)$$

$$-\frac{i}{2}\left(\bar{\psi}(-t,\vec{x})\left(-(-1)^\mu(-1)^\nu\sigma^\mu\sigma^\nu\right)\psi(-t,\vec{x})\right)$$

$$\mathcal{T}\bar{\psi}\sigma^{\mu\nu}\psi\mathcal{T} = -\frac{i}{2}(-1)^\mu(-1)^\nu\,\bar{\psi}(-t,\vec{x})\left[\sigma^\mu\sigma^\nu - \sigma^\nu\sigma^\mu\right]\psi(-t,\vec{x})$$

$$\boxed{\mathcal{T}\bar{\psi}\sigma^{\mu\nu}\psi\mathcal{T} = -(-1)^\mu(-1)^\nu\,\bar{\psi}(-t,\vec{x})\,\sigma^{\mu\nu}\,\psi(-t,\vec{x})}$$

Pseudoscalar Bilinear ($i\bar{\psi}\gamma^5\psi$)

$$iT\bar{\psi}\gamma^5\psi T = -i\bar{\psi}(-t,\vec{x})\gamma^1\gamma^3\gamma^5\gamma^1\gamma^3\psi(-t,\vec{x})$$

$$\gamma^1\gamma^3\gamma^5\gamma^1\gamma^3 = \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix}\begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}$$

$$= \begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & \sigma^1\sigma^3 \end{bmatrix}\begin{bmatrix} -\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^1\sigma^3\sigma^1\sigma^3 & 0 \\ 0 & -\sigma^1\sigma^3\sigma^1\sigma^3 \end{bmatrix}$$

$$\sigma^1\sigma^3\sigma^1\sigma^3 = -\overleftrightarrow{1}$$

$$= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= -\gamma^5$$

This one may not be correct.

$$\Rightarrow iT\bar{\psi}\gamma^5\psi T = i\bar{\psi}(-t,\vec{x})\gamma^5\psi(-t,\vec{x})$$

Charge Conjugation

Dirac fields transform under charge conjugation as

$$\begin{cases} C\psi(x)C = -i(\bar{\psi}\gamma^0\gamma^2)^T \\ C\bar{\psi}(x)C = -i(\gamma^0\gamma^2\psi)^T \end{cases}$$

Scalar Bilinear

$$\begin{aligned} C\bar{\psi}\psi C &= -(\gamma^0\gamma^2\psi)^T(\bar{\psi}\gamma^0\gamma^2)^T \\ &= -\left[(\bar{\psi}\gamma^0\gamma^2)(\gamma^0\gamma^2\psi)\right]^T \\ &= -\left[\bar{\psi}\gamma^0\gamma^2\gamma^0\gamma^2\psi\right]^T \\ &= -\left[-\bar{\psi}\gamma^2\gamma^0\gamma^0\gamma^2\psi\right]^T \\ &= -\left[-\bar{\psi}\psi\right]^T \\ &= \bar{\psi}\psi \end{aligned}$$

where in the last step $\bar{\psi}$ is a row vector, ψ is a column vector, and their matrix product is a scalar so the transpose has no effect.

Vector Charge Conjugation

$$C\bar{\psi}\gamma^\mu\psi C = -(\gamma^0\gamma^2\psi)^T\gamma^\mu(\bar{\psi}\gamma^0\gamma^2)^T$$

$$= -(\gamma^0\gamma^2\psi)^T\left[(\bar{\psi}\gamma^0\gamma^2)\gamma^{\mu T}\right]^T$$

$$= -\left[(\bar{\psi}\gamma^0\gamma^2)\gamma^{\mu T}(\gamma^0\gamma^2\psi)\right]^T$$

$$= -\left[\bar{\psi}\gamma^0\gamma^2\gamma^{\mu T}\gamma^0\gamma^2\psi\right]^T$$

Change this to employ the same technique as in the Tensor Bilinear

$$\gamma^0\gamma^2\gamma^{\mu T}\gamma^0\gamma^2 = \begin{bmatrix}0 & \sigma^0\\ -\sigma^0 & 0\end{bmatrix}\begin{bmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{bmatrix}\begin{bmatrix}0 & -\sigma^{\mu T}\\ \sigma^{\mu T} & 0\end{bmatrix}\begin{bmatrix}0 & \sigma^0\\ -\sigma^0 & 0\end{bmatrix}\begin{bmatrix}0 & \sigma^2\\ -\sigma^2 & 0\end{bmatrix}$$

$$= \begin{bmatrix}-\sigma^0\sigma^2 & 0\\ 0 & -\sigma^0\sigma^2\end{bmatrix}\begin{bmatrix}0 & -\sigma^{\mu T}\\ \sigma^{\mu T} & 0\end{bmatrix}\begin{bmatrix}-\sigma^0\sigma^2 & 0\\ 0 & -\sigma^0\sigma^2\end{bmatrix}$$

$$= \begin{bmatrix}0 & \sigma^0\sigma^2\sigma^{\mu T}\\ -\sigma^0\sigma^2\sigma^{\mu T} & 0\end{bmatrix}\begin{bmatrix}-\sigma^0\sigma^2 & 0\\ 0 & -\sigma^0\sigma^2\end{bmatrix}$$

$$= \begin{bmatrix}0 & -\sigma^0\sigma^2\sigma^{\mu T}\sigma^0\sigma^2\\ \sigma^0\sigma^2\sigma^{\mu T}\sigma^0\sigma^2 & 0\end{bmatrix}$$

$$\sigma^0\sigma^2\sigma^{\mu T}\sigma^0\sigma^2 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\sigma^{\mu T}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

$$= \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\sigma^{\mu T}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

If $\mu=0$,

$$\sigma^0\sigma^2\sigma^{\mu T}\sigma^0\sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \sigma^0$$

If $\mu=1$,

$$\sigma^0\sigma^2\sigma^{\mu T}\sigma^0\sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -\sigma^1$$

If $\mu=2$,

$$\sigma^0\sigma^2\sigma^{\mu T}\sigma^0\sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = -\sigma^2$$

If $\mu=3$,

$$\sigma^0\sigma^2\sigma^{\mu T}\sigma^0\sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = -\sigma^3$$

Take a step back and put these into $\gamma^0\gamma^2\gamma^{\mu T}\gamma^0\gamma^2$.

Wrong sign. Don't know why ←

If $\mu=0$,

$$\gamma^0\gamma^2\gamma^{\mu T}\gamma^0\gamma^2 = \begin{bmatrix} 0 & -\sigma^0 \\ \sigma^0 & 0 \end{bmatrix} = -\gamma^0$$

If $\mu=1$,

$$\gamma^0\gamma^2\gamma^{\mu T}\gamma^0\gamma^2 = \begin{bmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{bmatrix} = \gamma^1$$

If $\mu=2$, $\gamma^0\gamma^2\gamma^{\mu T}\gamma^0\gamma^2 = \begin{bmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{bmatrix} = \gamma^2$

If $\mu=3$, $\gamma^0\gamma^2\gamma^{\mu T}\gamma^0\gamma^2 = \begin{bmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{bmatrix} = \gamma^3$

$$(\bar{\psi}\gamma^\mu\psi)^\dagger = -\left[\bar{\psi}\gamma^\mu\psi\right]^T = -\bar{\psi}\gamma^\mu\psi$$

$$= -(\gamma^\mu\psi)^T\bar{\psi}^T$$

$$= -\psi^T\gamma^{\mu T}\bar{\psi}^T$$

(row) · (4×4) · (col.) =

= (row)(col.)

= scalar

Charge Conjugation, Tensor Bilinear:

$$C\bar{\psi}\sigma^{\mu\nu}\psi C = \frac{i}{2} C\bar{\psi}\gamma^\mu\gamma^\nu\psi C - \frac{i}{2} C\bar{\psi}\gamma^\nu\gamma^\mu\psi C$$

$$= \frac{i}{2}\left(-i\gamma^0\gamma^2\psi\right)^T\gamma^\mu\gamma^\nu\left(-i\bar{\psi}\gamma^0\gamma^2\right)^T - \frac{i}{2}\left(-i\gamma^0\gamma^2\psi\right)^T\gamma^\nu\gamma^\mu\left(-i\bar{\psi}\gamma^0\gamma^2\right)^T$$

$$= \frac{i}{2}\left[\gamma^{\mu T}\left(-i\gamma^0\gamma^2\psi\right)\right]^T\left[\left(-i\bar{\psi}\gamma^0\gamma^2\right)\gamma^{\nu T}\right]^T - \frac{i}{2}\left[\gamma^{\nu T}\left(-i\gamma^0\gamma^2\psi\right)\right]^T\left[\left(-i\bar{\psi}\gamma^0\gamma^2\right)\gamma^{\mu T}\right]^T$$

$$= \frac{i}{2}\left[\left(-i\bar{\psi}\gamma^0\gamma^2\right)\gamma^{\nu T}\gamma^{\mu T}\left(-i\gamma^0\gamma^2\psi\right)\right]^T - \frac{i}{2}\left[\left(-i\bar{\psi}\gamma^0\gamma^2\right)\gamma^{\mu T}\gamma^{\nu T}\left(-i\gamma^0\gamma^2\psi\right)\right]^T$$

$$= -\frac{i}{2}\left[\bar{\psi}\gamma^0\gamma^2\gamma^{\nu T}\gamma^{\mu T}\gamma^0\gamma^2\psi\right]^T + \frac{i}{2}\left[\bar{\psi}\gamma^0\gamma^2\gamma^{\mu T}\gamma^{\nu T}\gamma^0\gamma^2\psi\right]^T$$

$$\gamma^0\gamma^2\gamma^{\mu T}\gamma^{\nu T}\gamma^0\gamma^2 = \begin{bmatrix} -\sigma^0\sigma^2 & 0 \\ 0 & -\sigma^0\sigma^2 \end{bmatrix}\begin{bmatrix} -\sigma^{\mu T}\sigma^{\nu T} & 0 \\ 0 & -\sigma^{\mu T}\sigma^{\nu T} \end{bmatrix}\begin{bmatrix} -\sigma^0\sigma^2 & 0 \\ 0 & -\sigma^0\sigma^2 \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^0\sigma^2\sigma^{\mu T}\sigma^{\nu T} & 0 \\ 0 & \sigma^0\sigma^2\sigma^{\mu T}\sigma^{\nu T} \end{bmatrix}\begin{bmatrix} -\sigma^0\sigma^2 & 0 \\ 0 & -\sigma^0\sigma^2 \end{bmatrix}$$

$$= \begin{bmatrix} -\sigma^0\sigma^2\sigma^{\mu T}\sigma^{\nu T}\sigma^0\sigma^2 & 0 \\ 0 & -\sigma^0\sigma^2\sigma^{\mu T}\sigma^{\nu T}\sigma^0\sigma^2 \end{bmatrix}$$

I. If $\mu,\nu=\{0,2\}$, then $\gamma^\mu=\gamma^{\mu T}$.

$$\gamma^0\gamma^2\gamma^{\mu T}\gamma^{\nu T}\gamma^0\gamma^2 = -\gamma^0\gamma^\mu\gamma^2\gamma^\nu\gamma^0\gamma^2 = \gamma^0\gamma^2\gamma^\mu\gamma^\nu\gamma^0\gamma^2$$

$$= \gamma^0\gamma^\mu\gamma^\nu\gamma^2\gamma^0\gamma^2 = -\gamma^0\gamma^\mu\gamma^\nu\gamma^2\gamma^0\gamma^2$$

$$= \gamma^\mu\gamma^\nu\gamma^0\gamma^2\gamma^0\gamma^2 = \gamma^\mu\gamma^\nu\gamma^0\gamma^2\gamma^0\gamma^2$$

$$= -\gamma^\mu\gamma^\nu\gamma^0\gamma^0\gamma^2\gamma^2 = -\gamma^\mu\gamma^\nu\gamma^0\gamma^0\gamma^2\gamma^2$$

$$= -\gamma^\mu\gamma^\nu$$

II. If $\mu,\nu=\{1,3\}$, then $\gamma^\mu=-\gamma^{\mu T}$. However, the two negative signs will cancel:

$$\gamma^{\mu T}\gamma^{\nu T} = (-\gamma^\mu)(-\gamma^\nu) = \gamma^\mu\gamma^\nu$$

So we get the same result as for (I).

$$\gamma^0\gamma^2\gamma^{\mu T}\gamma^{\nu T}\gamma^0\gamma^2 = \gamma^0\gamma^2\gamma^\mu\gamma^\nu\gamma^0\gamma^2$$

$$= -\gamma^\mu\gamma^\nu$$

III. If $\mu=\{0,2\}$ and $\nu=\{1,3\}$, (or viceversa)

~~$\gamma^0\gamma^2\gamma^\mu\gamma^\nu\gamma^0$~~

$$\gamma^0\gamma^2\gamma^{\mu T}\gamma^{\nu T}\gamma^0\gamma^2 = -\gamma^0\gamma^2\gamma^\mu\gamma^\nu\gamma^0\gamma^2$$

This one is a little more tricky. One of μ or ν is 0 or 2. So one of γ^0 or γ^2 will anticommute with one of γ^μ or γ^ν; meanwhile the other of γ^0 or γ^2 will anticommute will both γ^μ and γ^ν. Summarily, three more negative signs are introduced in moving $\gamma^0\gamma^2$ to the right of $\gamma^\mu\gamma^\nu$.

$$\gamma^0\gamma^2\gamma^{\mu T}\gamma^{\nu T}\gamma^0\gamma^2 = \gamma^\mu\gamma^\nu\gamma^0\gamma^2\gamma^0\gamma^2$$

$$= -\gamma^\mu\gamma^\nu\gamma^0\gamma^0\gamma^2\gamma^2$$

$$= -\gamma^\mu\gamma^\nu$$

Cases (I), (II), and (III) all yield the same result.

$$\Rightarrow C\bar{\psi}\sigma^{\mu\nu}\psi C = -\frac{i}{2}\left[\bar{\psi}(-\gamma^\nu\gamma^\mu)\psi\right]^T + \frac{i}{2}\left[\bar{\psi}(-\gamma^\mu\gamma^\nu)\psi\right]^T$$

Within the bracket we have the matrix multiplication form: $\bar{\psi}\gamma^\nu\gamma^\mu\psi$

(row) $\underbrace{(4\times4)(4\times4)}_{(4\times4)}$ (column)

= $\underbrace{(\text{row})(4\times4)}_{(\text{row})}$ (column)

= (row)(column) = scalar

so the transpose now changes nothing.

$$C\bar{\psi}\sigma^{\mu\nu}\psi C = -\frac{i}{2}\bar{\psi}\gamma^\mu\gamma^\nu\psi + \frac{i}{2}\bar{\psi}\gamma^\nu\gamma^\mu\psi$$

$$= -\frac{i}{2}\bar{\psi}\left[\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right]\psi$$

$$\boxed{= -\bar{\psi}\sigma^{\mu\nu}\psi}$$