For the sake of performing a real, measurable calculation (cross section) from first principles (the QED Lagrangian), let's look at

$$e^+(p') + e^-(p) \longrightarrow \mu^+(k') + \mu^-(k)$$

I'm going to begin with the QED Lagrangian derive the scattering amplitude without using Feynman Diagrams, square the amplitude and average over spins, and finally obtain a formula for the cross section.

I'll then take a step back and compute the cross section for each set of spin orientations separately. Stated another way, I'll analyze the helicity structure.

Now to get started. I have a system with two distinct types of pairs of fermions, so the system Lagrangian density is the sum of two Dirac Lagrangians (allowing for EM interactions, of course).

$$\mathcal{L}(x) = \bar{\psi}_e(i\not{D} - m_e)\psi_e + \bar{\psi}_\mu(i\not{D} - m_\mu)\psi_\mu$$

$$\mathcal{L}(x) = \bar{\psi}_e(i\gamma^\rho\partial_\rho - m_e)\psi_e - e\bar{\psi}_e\gamma^\rho\psi_e A_\rho$$

$$+\bar{\psi}_\mu(i\gamma^\rho\partial_\rho - m_\mu)\psi_\mu - e\bar{\psi}_\mu\gamma^\rho\psi_\mu A_\rho$$

where the subscript "e" and "μ" designate electrons and muons just as labels, and "ρ" is the 4-vector script that is implicitly summed over.

The part I'm interested in is the interaction Lagrangian;

$$\mathcal{L}_I \equiv -e\bar{\Psi}_e \gamma^\rho \Psi_e A_\rho - e\bar{\Psi}_\mu \gamma^\rho \Psi_\mu A_\rho$$

From this I need the interaction Hamiltonian, which is in general defined as

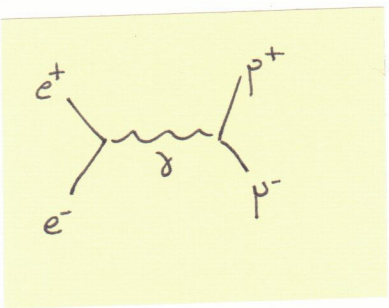
$$\mathcal{H}_I(x) = \pi(x)\dot{\Phi}(x) - \mathcal{L}_I(x)$$

where $\Phi(x)$ is one of the position-space fields and $\pi(x)$ is its conjugate momentum field. However, in this case there are no time-derivatives of fields so it is simply

$$\mathcal{H}_I(x) = -\mathcal{L}_I(x)$$

$$\mathcal{H}_I(x) = \underbrace{e\bar{\Psi}_e \gamma^\rho \Psi_e A_\rho}_{\equiv \mathcal{H}_{eI}(x)} + \underbrace{e\bar{\Psi}_\mu \gamma^\rho \Psi_\mu A_\rho}_{\equiv \mathcal{H}_{\mu I}(x)}$$

$$\mathcal{H}_I(x) = \mathcal{H}_{eI}(x) + \mathcal{H}_{\mu I}(x)$$



The scattering matrix for this interaction is

$$\langle k_1', k_2' | S | p_1, p_2 \rangle = \underbrace{\langle k_1', k_2' | \vec{1} | p_1', p_2' \rangle}_{\text{Trivial component}} + i \underbrace{\langle k_1', k_2' | T | p_1', p_2' \rangle}_{\text{Interaction component}}$$

The "interaction component" here is what I need to calculate. From eqn. (4.90),

$$i\langle k_1', k_2' | T | p_1', p_2' \rangle = \lim_{t \to \infty(1-i\varepsilon)} \left\langle k_1', k_2' \middle| T \exp\left[-i \int_{-t}^{t} dt' H_I(t')\right] \middle| p_1', p_2' \right\rangle$$

That exponential expands as (from eqn. 4.22),

$$T \exp\left[-i \int_{-t}^{t} dt' H_I(t')\right] = 1 + (-i) \int_{-t}^{t} dt_1 H_I(t_1) + \frac{(-i)^2}{2!} \int_{-t}^{t} dt_1 dt_2 T\left[H_I(t_1) H_I(t_2)\right] + \cdots$$

As we are beginning and ending with two particle states, the 2nd order term is the first that can contribute to this interaction, and any higher order terms contain loops that we aren't going to deal with here.

The scattering matrix becomes

$$\cong \langle k_1', k_2' | \frac{(-i)^2}{2!} \int\int dt_1 dt_2 T\left[H_I(x_1) H_I(x_2)\right] | p_1', p_2' \rangle$$

$$H_I(x) = \int d^3x \, \mathcal{H}_I(x)$$

$$= \int d^3x \left[\mathcal{H}_{Ie}(x) + \mathcal{H}_{Ip}(x)\right]$$

$$= \langle k_1' ; k_2 | \frac{(-i)^2}{2!} \iint dt_1 dt_2 \, T\left[\int d^3x_1 \, \mathcal{H}_I(x_1) \int d^3x_2 \, \mathcal{H}_I(x_2) \right] | p' ; p \rangle$$

$$= \langle k_1' ; k_2 | \frac{(-i)^2}{2!} T\left[\int d^4x_1 \, \mathcal{H}_I(x_1) \int d^4x_2 \, \mathcal{H}_I(x_2) \right] | p' ; p \rangle$$

$$= \frac{(-i)^2}{2} \iint d^4x_1 \, d^4x_2 \, \langle k_1' ; k_2 | T\left[\mathcal{H}_I(x_1) \mathcal{H}_I(x_2) \right] | p' ; p \rangle$$

$$= \frac{(-i)^2}{2} \iint d^4x_1 \, d^4x_2 \, \langle k_1' ; k_2 | T\left[\underset{I_e}{\mathcal{H}}(x_1) \underset{I_e}{\mathcal{H}}(x_2) + \underset{I_e}{\mathcal{H}}(x_1) \underset{I_\mu}{\mathcal{H}}(x_2) + \underset{I_\mu}{\mathcal{H}}(x_1) \underset{I_e}{\mathcal{H}}(x_2) + \underset{I_\mu}{\mathcal{H}}(x_1) \underset{I_\mu}{\mathcal{H}}(x_2) \right] | p' ; p \rangle$$

Since I have an interaction that includes both electrons and muons, the $\mathcal{H}_{I_e} \mathcal{H}_{I_e}$ and $\mathcal{H}_{I_\mu} \mathcal{H}_{I_\mu}$ terms do not contribute.

$$= \frac{(-i)^2}{2} \iint d^4x_1 \, d^4x_2 \, \langle k_1' ; k_2 | T\left[\underset{I_e}{\mathcal{H}}(x_1) \underset{I_\mu}{\mathcal{H}}(x_2) + \underset{I_\mu}{\mathcal{H}}(x_1) \underset{I_e}{\mathcal{H}}(x_2) \right] | p' ; p \rangle$$

Next, I have to calculate these two time-ordered products.

$$T\left[\mathcal{H}_{I_e}(x_1) \mathcal{H}_{I_\mu}(x_2) \right] = e^2 T\left[\bar{\psi}_e(x_1) \gamma^\rho \psi_e(x_1) A_\rho(x_1) \bar{\psi}_\mu(x_2) \gamma^\rho \psi_\mu(x_2) A_\rho(x_2) \right]$$

For brevity, I'm going to drop some labels.

$$= e^2 N\big[\bar{\psi}_1 \gamma^\rho \psi_1 A_{1\rho}\, \bar{\psi}_2 \gamma^\rho \psi_2 A_{2\rho} + \text{contractions}\big]$$

$$= e^2 N\big[\bar{\psi}_1 \gamma^\rho \psi_1 A_{1\rho}\, \bar{\psi}_2 \gamma^\rho \psi_2 A_{2\rho} + \overbrace{\bar{\psi}_1 \gamma^\rho \psi_1} A_{1\rho}\, \bar{\psi}_2 \gamma^\sigma \psi_2 A_{2\sigma}$$
$$+ \bar{\psi}_1 \gamma^\rho \psi_1 A_{1\rho}\, \overbrace{\bar{\psi}_2 \gamma^\sigma \psi_2} A_{2\sigma} + \bar{\psi}_1 \gamma^\rho \psi_1 A_{1\rho}\, \bar{\psi}_2 \gamma^\sigma \psi_2 A_{2\sigma}$$
$$+ \bar{\psi}_1 \gamma^\rho \psi_1 A_{1\rho}\, \bar{\psi}_2 \gamma^\sigma \psi_2 A_{2\sigma} + \bar{\psi}_1 \gamma^\rho \psi_1 A_{1\rho}\, \bar{\psi}_2 \gamma^\sigma \psi_2 A_{2\sigma}$$
$$+ \bar{\psi}_1 \gamma^\rho \overbrace{\psi_1 A_{1\rho}\, \bar{\psi}_2 \gamma^\sigma} \psi_2 A_{2\sigma} + \bar{\psi}_1 \gamma^\rho \psi_1 A_{1\rho}\, \bar{\psi}_2 \gamma^\sigma \psi_2 A_{2\sigma}$$
$$+ \bar{\psi}_1 \gamma^\rho \overbrace{\psi_1 A_{1\rho}\, \bar{\psi}_2 \gamma^\sigma \psi_2} A_{2\sigma} + \bar{\psi}_1 \gamma^\rho \psi_1 A_{1\rho}\, \bar{\psi}_2 \gamma^\sigma \psi_2 A_{2\sigma}$$
$$+ \bar{\psi}_1 \gamma^\rho \psi_1 A_{1\rho}\, \overbrace{\bar{\psi}_2 \gamma^\sigma \psi_2 A_{2\sigma}} + \bar{\psi}_1 \gamma^\rho \psi_1 A_{1\rho}\, \bar{\psi}_2 \gamma^\sigma \psi_2 A_{2\sigma}$$
$$+ \bar{\psi}_1 \gamma^\rho \psi_1 A_{1\rho}\, \overbrace{\bar{\psi}_2 \gamma^\sigma \psi_2 A_{2\sigma}} + (\text{terms w/ more than one contraction})\big]$$

- Contractions b/w fields of different types vanish.
- If there are no contractions of a field at x_1 with a field at x_2, then there is no interaction between the initial particles.
- As the initial states and final states are all fermions, the terms that contribute to this interaction must have the four fermionic fields uncontracted so that they are free to contract with the momentum state vectors.

Hence, the only term left is

$$e^2 T\left[\bar{\Psi}_1 \gamma^\rho \Psi_1 A_{1\rho}\, \bar{\Psi}_2 \gamma^\sigma \Psi_2 A_{2\sigma}\right] = e^2 \overline{\left[\bar{\Psi}_1 \gamma^\rho \Psi_1 A_{1\rho}\, \bar{\Psi}_2 \gamma^\sigma \Psi_2 A_{2\sigma}\right]}$$

Put this back into the scattering matrix element for both terms.

$$\frac{(-ie)^2}{2}\iint d^4x_1 d^4x_2 \left[\langle k_1' i k_2' | T\left[\mathcal{H}_{Ie}(x_1)\mathcal{H}_{Ip}(x_2)\right]|p' i p\rangle + \langle k_1' i k_2' | T\left[\mathcal{H}_{Ip}(x_1)\mathcal{H}_{Ie}(x_2)\right]|p' i p\rangle\right]$$

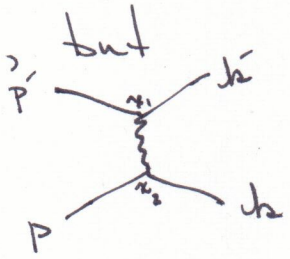
$$= \frac{(-ie)^2}{2}\iint d^4x_1 d^4x_2 \langle k_1' i k_2' | \overline{\left[\bar{\Psi}_e(x_1)\gamma^\rho \Psi_e(x_1) A_\rho(x_1)\, \bar{\Psi}_\mu(x_2)\gamma^\sigma \Psi_\mu(x_2) A_\rho(x_2)\right]} |p' i p\rangle$$

$$+ \frac{(-ie)^2}{2}\iint d^4x_1 d^4x_2 \langle k_1' i k_2' | \overline{\left[\bar{\Psi}_\mu(x_1)\gamma^\rho \Psi_\mu(x_1) A_\rho(x_1)\, \bar{\Psi}_e(x_2)\gamma^\sigma \Psi_e(x_2) A_\sigma(x_2)\right]} |p' i p\rangle$$

This amplitude still covers several kinds of interactions, unless we understand that $|p' i p\rangle$ is implicitly an electron state so only the electron wavefunctions are eigenfunctions of it. Ditto for the muon state vector $\langle k_1' i k_2'|$. This means that

(s-channel)

is allowed, but



(t-channel)

is not.

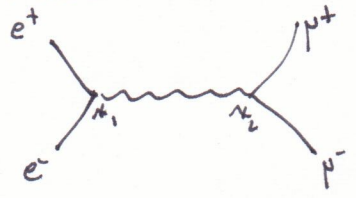
It is important to note that if the end-states were also electrons, then both interactions are allowed.

Fully contracted, the amplitude is

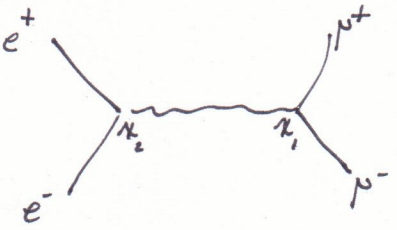
$$\frac{(-ie^*)^2}{2}\iint d^4x_1 d^4x_2 \langle k_1 k_2 | \bar{\psi}_e(x_1)\gamma^\rho \psi_e(x_1) A_\rho(x_1) \bar{\psi}_\mu(x_2)\gamma^\sigma \psi_\mu(x_2) A_\sigma(x_2) | p' p\rangle$$

$$+\frac{(-ie^*)^2}{2}\iint d^4x_1 d^4x_2 \langle k_1 k_2 | \bar{\psi}_\mu(x_1)\gamma^\rho \psi_\mu(x_1) A_\rho(x_1) \bar{\psi}_e(x_2)\gamma^\sigma \psi_e(x_2) A_\sigma(x_2) | p' p\rangle$$

These two terms cover



and



respectively.

$$= \frac{(-ie^*)^2}{2}\iint d^4x_1 d^4x_2\, e^{-ip'\cdot x_1}\bar{v}(p')\gamma^\rho e^{-ip\cdot x_1}u(p)\int\frac{d^4q}{(2\pi)^4}\left[\frac{-ig_{\rho\sigma}}{q^2+i\varepsilon}\right]e^{-iq\cdot(x_2-x_1)}e^{ik_2\cdot x_2}\bar{u}(k_2)\gamma^\sigma e^{ik_1'\cdot x_2}v(k_1')$$

$$+\frac{(-ie)^2}{2}\iint d^4x_1 d^4x_2\, e^{ik_1'\cdot x_1}\bar{u}(k_1')\gamma^\rho e^{ik_2'\cdot x_1}v(k_2')\int\frac{d^4q}{(2\pi)^4}\left[\frac{-ig_{\rho\sigma}}{q^2+i\varepsilon}\right]e^{-iq\cdot(x_1-x_2)}e^{-ip'\cdot x_2}\bar{v}(p')\gamma^\sigma e^{-ip\cdot x_2}u(p)$$

$$= \frac{(-ie)^2}{2}\iint d^4x_1 d^4x_2\,\bar{v}(p')\gamma^\rho u(p)\, e^{-i(p'+p-q)\cdot x_1}\int\frac{d^4q}{(2\pi)^4}\left[\frac{-ig_{\rho\sigma}}{q^2+i\varepsilon}\right]\boxed{\bar{u}(k_2)\gamma^\sigma v(k_1')}\, e^{-i(q-k_2-k_1')\cdot x_2}$$

$$+\frac{(-ie)^2}{2}\iint d^4x_1 d^4x_2\,\bar{u}(k_2)\gamma^\rho v(k_1')\, e^{-i(-k_2-k_1'+q)\cdot x_1}\int\frac{d^4q}{(2\pi)^4}\left[\frac{-ig_{\rho\sigma}}{q^2+i\varepsilon}\right]\bar{v}(p')\gamma^\sigma u(p)\, e^{-i(-q+p'+p)\cdot x_2}$$

$$= \frac{(-ie)^2}{2}\bar{v}(p')\gamma^\rho u(p)(2\pi)^4\delta^{(4)}(p'+p-q)\int\frac{d^4q}{(2\pi)^4}\left[\frac{-ig_{\rho\sigma}}{q^2+i\varepsilon}\right]\bar{u}(k)\gamma^\sigma v(k')(2\pi)^4\delta^{(4)}(q-k-k')$$

$$+\frac{(-ie)^2}{2}\bar{u}(k)\gamma^\rho v(k')(2\pi)^4\delta^{(4)}(q-k-k')\int\frac{d^4q}{(2\pi)^4}\left[\frac{-ig_{\rho\sigma}}{q^2+i\varepsilon}\right]\bar{v}(p')\gamma^\sigma u(p)(2\pi)^4\delta^{(4)}(p'+p-q)$$

$$= \frac{ie^2}{2}\int d^4q\,\bar{v}(p')\gamma^\rho u(p)\left[\frac{g_{\rho\sigma}}{q^2}\right]\bar{u}(k)\gamma^\sigma v(k')(2\pi)^4\delta^{(4)}(p'+p-q)\delta^{(4)}(q-k-k')$$

$$+\frac{ie^2}{2}\int d^4q\,\bar{u}(k)\gamma^\rho v(k')\left[\frac{g_{\rho\sigma}}{q^2}\right]\bar{v}(p')\gamma^\sigma u(p)(2\pi)^4\delta^{(4)}(p'+p-q)\delta^{(4)}(q-k-k')$$

$$= \frac{ie^2}{2}(2\pi)^4\bar{v}(p')\gamma^\rho u(p)\left[\frac{g_{\rho\sigma}}{q^2}\right]\bar{u}(k)\gamma^\sigma v(k')$$

$$+\frac{ie^2}{2}(2\pi)^4\bar{u}(k)\gamma^\rho v(k')\left[\frac{g_{\rho\sigma}}{q^2}\right]\bar{v}(p')\gamma^\sigma u(p)$$

$$= \frac{ie^2(2\pi)^4}{q^2}\bar{v}(p')\gamma^\rho u(p)\,\bar{u}(k)\gamma_\rho v(k')$$

Hence, the invariant amplitude is

$$\boxed{iM = \frac{ie^2}{q^2}\bar{v}(p')\gamma^\rho u(p)\,\bar{u}(k)\gamma_\rho v(k')}$$

where $q = k+k' = p+p'$ explicitly b/c of the integration over the δ-functions

5.1 $e^+e^- \to \mu^+\mu^-$: Introduction

The next step in finding the differential cross section is to compute $|M|^2$. This means finding the complex conjugate M^*. Observe the following:

$$\begin{aligned}(\bar{v}\gamma^\nu u)^* &= (v^\dagger \gamma^0 \gamma^\nu u)^* \\ &= (\gamma^\nu u)^* (v^\dagger \gamma^0)^* \\ &= u^\dagger (\gamma^\nu)^\dagger (\gamma^0)^\dagger v \\ &= u^\dagger (\gamma^\nu)^\dagger \gamma^0 v \\ &= u^\dagger \gamma^0 \gamma^\nu v \\ &= \bar{u}\gamma^\nu v\end{aligned}$$

This is convenient since both sides appear in the amplitude already anyhow.

$$|M|^2 = \frac{e^2}{q^2}\left[\bar{v}(p')\gamma^\mu u(p)\right]\left[\bar{u}(p)\gamma^\nu v(p')\right]\left[\bar{u}(k)\gamma_\mu v(k')\right]\left[\bar{v}(k')\gamma_\nu u(k)\right]$$

Denote the electron spin states by s, s' and the muon spin states r, r'. In most experiments, the electron beams are unpolarized so we <u>average</u> over spins s, s'. Muon detectors are blind to polarization so the cross section is a sum of the possible polarization states.

We want to compute $\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'}\sum_r\sum_{r'} |M(s,s'\to r,r')|^2$.

To do this, we compute the above quantity by using the spin sums (3.66) and (3.67).

$$\frac{1}{4}\sum_s\sum_{s'}\sum_r\sum_{r'}|M|^2$$

$$= \frac{1}{4}\frac{e^4}{q^4}\sum_{s,s',r,r'}\left[\bar{v}^s(p')\gamma^\mu u^{s'}(p)\bar{u}^{s'}(p)\gamma^\nu v^s(p')\right]\left[\bar{u}^r(k)\gamma_\mu v^{r'}(k')\bar{v}^{r'}(k')\gamma_\nu u^r(k)\right]$$

I need to commute $\bar{v}^s(p')$ just to the right of $v^s(p')$ to use the spin sum. Notice how $\bar{v}\gamma^\mu u$ is a row vector times a 4×4 matrix (yielding another row vector) times a column vector (yielding a scalar sum of many terms). To proceed, I'll write the same thing in terms of individual elements rather than matrices. Assume repeated indices are summed over.

$$= \frac{e^4}{4q^4}\sum_{s,s',r,r'}\left[\bar{v}^s_a(p')\gamma^\mu_{ab}u^{s'}_b(p)\bar{u}^{s'}_c(p)\gamma^\nu_{cd}v^s_d(p')\right]\left[\bar{u}^r_i(k)(\gamma_\mu)_{ij}v^{r'}_j(k')\bar{v}^{r'}_k(k')(\gamma_\nu)_{kl}u^r_l(k)\right]$$

$$= \frac{e^4}{4q^4}\sum_{s,s',r,r'}\left[v^s_d(p')\bar{v}^s_a(p')\gamma^\mu_{ab}u^{s'}_b(p)\bar{u}^{s'}_c(p)\gamma^\nu_{cd}\right]\left[u^r_j(k)\bar{u}^r_i(k)(\gamma_\mu)_{ij}v^{r'}_j(k')\bar{v}^{r'}_k(k')(\gamma_\nu)_{kl}\right]$$

Now the spin sums can be utilized. Summing over s will pick out the "da" element of $(\not{p}' - m)$; summing over s' will pick out the "bc" element of $(\not{p} + m)$, and so on.

$$= \frac{e^4}{4q^2} \Big[\Big[(\not{p}' - m_e)_{da} \gamma^\mu_{ab} (\not{p} + m_e)_{bc} \gamma^\nu_{cd} \Big] \Big[(\not{k} + m_\mu)_{if} (\gamma_\mu)_{fg} (\not{k}' - m_\mu)_{gi} (\gamma_\nu)_{ij} \Big] \Big]$$

Next, $(\not{p}' - m_e)_{da}$ is a 4×4 matrix, as is γ^μ_{ab}. So the matrix product of them $(\not{p}' - m_e)\gamma^\mu$ is also a 4×4 matrix. However, also remember that what we are actually looking at here is $\sum_a (\not{p}' - m_e)_{da} \gamma^\mu_{ab}$. If it were the case that $d=b$, then this would be $\sum_{a,b} (\not{p}' - m_e)_{ba} \gamma^\mu_{ab}$.

This additional restriction of $d=b$ means that when we look at the $[(\not{p}' - m_e)\gamma^\mu]_{db}$ matrix we have the sum of the elements on the diagonal, or the "Trace." Generalizing this concept to the

$$= \frac{e^4}{4q^2} \Big[\Big[(\not{p}' - m_e)\gamma^\mu \Big]_{db} \Big[(\not{p} + m_e)\gamma^\nu \Big]_{bd} \Big[\Big[(\not{k} + m_\mu)\gamma_\mu \Big]_{ig} \Big[(\not{k}' - m_\mu)\gamma_\nu \Big]_{gi} \Big]$$

$$= \frac{e^4}{4q^2} \mathrm{Tr}\Big[(\not{p}' - m_e)\gamma^\mu (\not{p} + m_e)\gamma^\nu \Big] \mathrm{Tr}\Big[(\not{k} + m_\mu)\gamma_\mu (\not{k}' - m_\mu)\gamma_\nu \Big]$$

Despite the fact we work the expression down to the product of two traces, notice how each trace is still just one component of a rank-2 spacetime tensor, given by the '$\mu\nu$' indices! Denote them for our purposes here as the "Electron Tensor" and the "Muon Tensor." Both are solved in the same manner, collectively as "Lepton Tensors."

$$\mathrm{Tr}\left[(\not{k}+m)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\right] = \mathrm{Tr}\left[(\not{k}\gamma_\mu + m\gamma_\mu)(\not{k}'\gamma_\nu - m_\mu\gamma_\nu)\right]$$

$$= \mathrm{Tr}\left[\not{k}\gamma_\mu\not{k}'\gamma_\nu - \not{k}\gamma_\mu m\gamma_\nu + m\gamma_\mu\not{k}'\gamma_\nu - m\gamma_\mu m\gamma_\nu\right]$$

$$= \underbrace{\mathrm{Tr}\left[\not{k}\gamma_\mu\not{k}'\gamma_\nu\right]}_{4\left[k'_\mu k_\nu + k'_\nu k_\mu - k\cdot k' g_{\mu\nu}\right]} - m\underbrace{\mathrm{Tr}\left[\not{k}\gamma_\mu\gamma_\nu\right]}_{=0} + m\underbrace{\mathrm{Tr}\left[\gamma_\mu\not{k}'\gamma_\nu\right]}_{=0} - m^2\underbrace{\mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right]}_{=4g_{\mu\nu}}$$

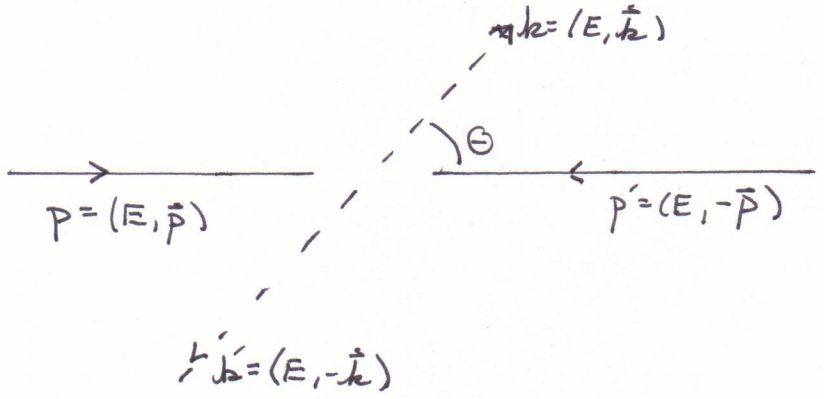
$$= 4\left[k_\mu k'_\nu + k_\nu k'_\mu - g_{\mu\nu}(k\cdot k' + m^2)\right]$$

Using this,

$$\Rightarrow \frac{1}{4}\sum_{s,s',r,r'}|M|^2 = \frac{8e^4}{q^4}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')\right]$$

where $m_e \approx 0$ as an approximation.

So far, we have done without choosing a particular frame of reference. To proceed, consider the center of momentum frame.



$$E^2 = p^2 + m_e^2$$

$$E \approx |\vec{p}|$$

$$\Rightarrow p = (E, p_z = E\hat{z}) \quad , \quad p' = (E, p_z' = -E\hat{z})$$

$$\underline{q^2 = (p+p')^2 = 4E^2}$$

$$\underline{p \cdot p' = 2E^2}$$

$$\underline{p \cdot k = p' \cdot k' = E^2 - E|\vec{k}|\cos\theta}$$

$$\underline{p \cdot k' = p' \cdot k = E^2 + |\vec{k}|\cos\theta}$$

$$\Rightarrow \frac{1}{4}\sum_{\text{spins}} |M|^2 = e^4 \left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta \right]$$

And finally, for the differential cross section use (4.84)

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{1}{2E_A 2E_B |v_A - v_B|} \frac{|\vec{p}|}{(2\pi)^2 4E_{cm}} |M|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_{cm}^2} \frac{|\vec{p}_2|}{16\pi^2 E_{CM}} \frac{1}{4} \sum |M|^2$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{CM}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2\theta \right]}$$