Spontaneously Broken Global U(1) Symmetry

Consider the $\mathbb{C}$ scalar

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

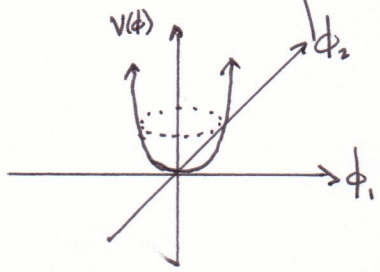
$$\phi^\dagger = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

and its Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \frac{1}{4}\lambda(\phi^\dagger\phi)^2$$

If $\mu^2 > 0$, then this describes a complex scalar particle pair with mass μ. The term $\mu^2\phi^\dagger\phi$ is the potential energy associated with their rest mass, or the energy of simply existing; and $\frac{1}{4}\lambda(\phi^\dagger\phi)^2$ is their kinetic and interaction energies.

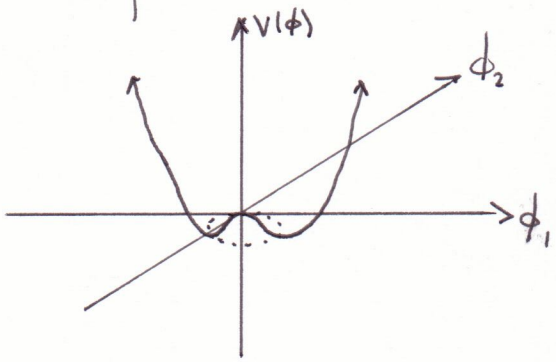
The potential $V(\phi) = \mu^2\phi^\dagger\phi + \frac{1}{4}\lambda(\phi^\dagger\phi)^2$ looks like



and finding the particle spectrum means calculating perturbative oscillations from the minimum of $V(\phi)$.

This Lagrangian is certainly invariant to a global U(1) field transformation: $\phi \to \phi' = e^{i\alpha}\phi$, but not a local transformation: $\phi \to \phi' = e^{i\alpha(x)}\phi$. We know experimentally that gauge bosons exist and their introduction (w/ nonzero masses) requires local gauge invariance. Spontaneous Symmetry breaking allows a massive and nondegenerate particle spectrum.

Spontaneous symmetry breaking happens for the case of $\mu^2 < 0$. Now the potential looks like



with an entire minima circle. We must choose a point on this circle to perform a perturbative expansion and find the particle spectrum. The minima manifold is found at

$$\frac{dV}{d(\phi^\dagger\phi)} = \mu^2 + \frac{\lambda}{4} 2\phi^\dagger\phi = 0$$

$$\phi^\dagger\phi = -\frac{2\mu^2}{\lambda}$$

$$\cancel{\phi_1^2 + \phi_2^2 = 4\mu^2}$$

$$\frac{1}{2}(\phi_1^2 + \phi_2^2) = -\frac{2\mu^2}{\lambda}$$

$$\phi_1^2 + \phi_2^2 = \frac{-4\mu^2}{\lambda} \equiv v^2$$

The new parameter v is then the radius of the circle.

One way to proceed is to expand about the point $\phi_1 = v$, $\phi_2 = 0$. Let

$$\phi(x) = \frac{1}{\sqrt{2}}\left[v + \eta(x) + i\xi(x)\right]$$

Then $\eta(x)$ is a field perturbation in the $\mathbb{R}$ direction and the perturbation $\xi(x)$ is in the purely imaginary direction. We find the consequent particle spectrum by putting this expression of $\phi(x)$ back into the Lagrangian.

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \frac{1}{4}\lambda (\phi^\dagger \phi)^2$$

$$= \frac{1}{2}(v + \partial_\mu \eta - i\partial_\mu \xi)(v + \partial^\mu \eta + i\partial^\mu \xi) - \frac{\mu^2}{2}(v + \eta - i\xi)(v + \eta + i\xi)$$
$$- \frac{1}{16}\lambda\left[(v+\eta - i\xi)(v+\eta+i\xi)\right]^2$$

$$= \frac{1}{2}\Big(v^2 + v(\partial^\mu \eta) + iv(\partial^\mu \xi) + (\partial_\mu \eta)v + (\partial_\mu \eta)(\partial^\mu \eta) + i(\partial_\mu \eta)(\partial^\mu \xi)$$
$$-i(\partial_\mu \xi)v - i(\partial_\mu \xi)(\partial^\mu \eta) + (\partial_\mu \xi)(\partial^\mu \xi)\Big) - \frac{\mu^2}{2}\Big(v^2 + v\eta + iv\xi$$
$$+ v\eta + \eta^2 + i\eta\xi - iv\xi - i\eta\xi + \xi^2\Big) - \frac{\lambda}{16}\Big(v^2 + v\eta + iv\xi$$
$$-i(\partial_\mu \xi)v - i(\partial_\mu \xi)(\partial^\mu \eta) + (\partial_\mu \xi)(\partial^\mu \xi)\Big)^2$$

$$\vdots$$

$$\Rightarrow \mathcal{L} = \frac{1}{2}(\partial_\mu \xi)^2 + \frac{1}{2}(\partial_\mu \eta)^2 + \mu^2 \eta^2 + (\text{cubic and quartic terms})$$

Comparing "$\mu^2 \eta^2$" to "$-\frac{1}{2}m_\eta^2 \eta$" we see the η-field perturbation is associated with a particle of mass $m_\eta = \sqrt{-2\mu^2}$. There is no mass term for the ξ-field. In attempting to generate a massive gauge boson, spontaneously broken gauge theory has it's own massless particle.

Pictorially, notice that the η-perturbation climbs up the potential well, while the ξ-perturbation moves ~~along~~ tangent to the ~~minimum~~ minima manifold. As the ξ-perturbation does not climb the potential, it does not acquire mass.

This suggests a possible more appropriate choice of how to parametrize the field perturbations. Remember that the choice to expand about $\phi_1 = v$, $\phi_2 = 0$ was arbitrary. Let's parametrize the perturbations in polar, rather than Cartesian, coordinates. That way we need not specify where on the minima manifold we expand around; the argument applies to all choices. One field perturbation will be in the radial direction, the other in the angular.

$$\phi(x) = \underbrace{\frac{\rho(x)}{\sqrt{2}}}_{\text{radial perturbation}} e^{\overbrace{\frac{i}{v}\theta(x)}^{\text{angular perturbation}}}$$

where since the minima-manifold has radius v, $\rho(x) = v + h(x)$

Just as before, we put $\phi(x)$ back into the Lagrangian and see what the particle spectrum looks like.

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \frac{1}{4}\lambda (\phi^\dagger \phi)^2$$

$$\partial_\mu \phi = \frac{1}{\sqrt{2}} \partial_\mu \rho(x) e^{\frac{i}{v}\theta(x)} + \frac{1}{\sqrt{2}} \rho(x) \frac{i}{v} \partial_\mu \theta(x) e^{\frac{i}{v}\theta(x)}$$

$$= \frac{1}{\sqrt{2}} \left[\partial_\mu \rho(x) e^{\frac{i}{v}\theta(x)} + \frac{i\rho(x)}{v} \partial_\mu \theta(x) e^{\frac{i}{v}\theta(x)} \right]$$

$$= \frac{1}{\sqrt{2}} e^{\frac{i}{v}\theta(x)} \left[\partial_\mu \rho(x) + \frac{i}{v} \rho(x) \partial_\mu \theta(x) \right]$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \rho(x) - \frac{i}{v}\rho(x)\partial_\mu\theta(x)\right)\left(\partial^\mu \rho(x) + \frac{i}{v}\rho(x)\partial^\mu\theta(x)\right)$$

$$- \frac{1}{2}\mu^2 \rho^2(x) - \frac{1}{16}\lambda \rho^4(x)$$

$$= \frac{1}{2}\left((\partial_\mu\rho)(\partial^\mu\rho) + \cancel{\frac{i}{v}\rho(\partial_\mu\rho)(\partial^\mu\theta)} - \cancel{\frac{i}{v}\rho(\partial_\mu\theta)(\partial^\mu\rho)} + \frac{1}{v^2}\rho^2(\partial_\mu\theta)(\partial^\mu\theta)\right)$$

$$- \frac{1}{2}\mu^2\rho^2 - \frac{1}{16}\lambda\rho^4$$

$$= \frac{1}{2}(\partial_\mu h + v)(\partial^\mu h + v) + \frac{1}{2v^2}(h+v)^2(\partial_\mu\theta)^2 - \frac{1}{2}\mu^2(h^2 + 2vh + v^2)$$

$$- \frac{1}{16}\lambda(h+v)^4$$

$$= \frac{1}{2}(\partial_\mu h)^2 + v(\partial_\mu h) + \frac{1}{2}v^2 + \left(\frac{1}{2v^2}h^2 + \frac{1}{v^2}hv + \frac{1}{2v^2}v^2\right)(\partial_\mu\theta)^2$$

$$- \frac{1}{2}\mu^2 h^2 - \mu^2 v h - \frac{1}{2}\mu^2 v^2 - \frac{1}{16}\lambda(h+v)^4$$

$$= \frac{1}{2}(\partial_\mu h)^2 + \frac{1}{2}(\partial_\mu\theta)^2 - \frac{1}{2}\mu^2 h^2 + \dots$$

Hence, we have kinetic terms for both $h(x)$ and $\theta(x)$, but a mass term only for $h(x)$. Also, given this parametrization of $\phi(x)$, the vacuum expectation value is

$$\langle 0|\phi|0\rangle = \frac{v}{\sqrt{2}}$$