U(1) Local Spontaneous Symmetry Breaking

Recall the globally gauge invariant Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2$$

To make this Lagrangian invariant to local gauge transformat
we must replace the derivatives by "covariant derivatives".

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$$

and include a kinetic term for the "gauge field" A_μ.

$$\mathcal{L} = \left[(\partial^\mu + iqA^\mu)\phi\right]^\dagger \left[(\partial_\mu + iqA_\mu)\phi\right] - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4}\lambda (\phi^\dagger \phi)^2 - \mu^2 (\phi^\dagger \phi)$$

This Lagrangian is then invariant to a local U(1) field transformation

$$\phi(x) \to \phi'(x) = e^{-i\alpha(x)} \phi(x)$$

or, in infinitesimal form

$$\phi(x) \to \phi'(x) = (1 - i\alpha(x))\phi(x)$$

We still do not know how the gauge field itself transforms. The point of the covariant derivative is to have $D^\mu \phi$ transform the same way ϕ does. So assume

$$D'^\mu \phi' = (1 - i\alpha(x)) D^\mu \phi$$

to be true and derive the transformation law for A_μ from it.

$$(\partial^\mu + iqA'^\mu)\phi' = (1 - i\alpha(x))(\partial^\mu + iqA^\mu)\phi$$

Since this is an infinitesimal transformation, the transformation of A^μ should have the form $A^\mu \to A'^\mu = A^\mu + \delta A^\mu$.

$$(\partial^\mu + iqA^\mu + iq\,\delta A^\mu)(1 - i\alpha(x))\phi = (1 - i\alpha(x))(\partial^\mu + iqA^\mu)\phi$$

$$(\cancel{\partial^\mu} - i\partial^\mu\alpha(x) + \cancel{iqA^\mu} + \cancel{qA^\mu\alpha(x)} + iq\,\delta A^\mu + \cancel{q\,\delta A^\mu\alpha(x)})\phi = (\cancel{\partial^\mu} + \cancel{iqA^\mu} - i\alpha(x)\partial^\mu + \cancel{\alpha(x)qA^\mu})\phi$$

small

Note that both $\alpha(x)$ and δA^μ are infinitesimals, so any terms infinitesimal to 2nd order or higher drop.

$$(-i\partial^\mu\alpha(x) + iq\,\delta A^\mu)\phi = (-i\alpha(x)\,\partial^\mu)\phi$$

$$iq\,\delta A^\mu\phi = (i\partial^\mu\alpha(x) - i\alpha(x)\,\partial^\mu)\phi$$

$$q\,\delta A^\mu\phi = \partial^\mu(\alpha\phi) - \alpha(\partial^\mu\phi)$$

$$q\,\delta A^\mu\phi = (\partial^\mu\alpha)\phi$$

$$\delta A^\mu = \frac{1}{q}\partial^\mu\alpha$$

Hence, the gauge field transforms as

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \frac{1}{q}\partial^\mu\alpha(x)$$

Now we'll see how the gauge fields A^μ absorbs the massless Goldstone bosons $\vec{\Theta}$ that were present in the global case.

Let's study the field equation for A^μ.

$$\begin{cases} \Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\mu_{em} \\ j^\nu_{em} = iq\left(\phi^\dagger(\partial^\nu\phi) - (\partial^\nu\phi)^\dagger\phi\right) - 2q^2 A^\nu \phi^\dagger\phi \end{cases}$$

Now recall the U(1) field parametrization for spontaneous symmetry breaking.

$$\phi(x) = \frac{1}{\sqrt{2}}(v + h(x))\, e^{-\frac{i}{v}\theta(x)}$$

The current becomes

$$j^\nu_{em} = iq\Big[\frac{1}{\sqrt{2}}(v+h)e^{\frac{i}{v}\theta(x)}\Big(\Big(\frac{1}{\sqrt{2}}(v+\partial^\nu h)e^{-\frac{i}{v}\theta(x)}\Big) + \frac{1}{\sqrt{2}}(v+h)\Big(-\frac{i}{v}\partial^\nu\theta\Big)e^{-\frac{i}{v}\theta(x)}$$

$$-\Big(\frac{1}{\sqrt{2}}(v+\partial^\nu h)e^{\frac{i}{v}\theta(x)} + \frac{1}{\sqrt{2}}(v+h)\frac{i}{v}\partial^\nu\theta\, e^{\frac{i}{v}\theta(x)}\Big)\frac{1}{\sqrt{2}}(v+h)e^{-\frac{i}{v}\theta(x)}\Big]$$

$$-2q^2 A^\nu \cdot \frac{1}{2}(v+h)^2$$

$$j^\nu_{em} = -\frac{iq}{2}\Big[(v+h)(v+\partial^\nu h) + (v+h)^2\Big(-\frac{i}{v}\partial^\nu\theta\Big)$$
$$-(v+\partial^\nu h)(v+h) - (v+h)^2\Big(\frac{i}{v}\partial^\nu\theta\Big)\Big]$$
$$-q^2 A^\nu (v+h)^2$$

$$j^\nu_{em} = -\frac{iq}{2}\Big[\frac{2i}{v}(v+h)^2\partial^\nu\theta\Big] - q^2 A^\nu (v+h)^2$$

$$j^\nu_{em} = \frac{q}{v}(v+h)^2\partial^\nu\theta - q^2 A^\nu (v+h)^2$$

$$j^\nu_{em} = \frac{q}{v}(v^2+2vh+h^2)\partial^\nu\theta - q^2A^\nu(v^2+2vh+h^2)$$

$$= qv\,\partial^\nu\theta + 2qh\,\partial^\nu\theta + \frac{q}{v}h^2\,\partial^\nu\theta - q^2A^\nu v^2 - 2vq^2A^\nu h - q^2A^\nu h^2$$

$$= -v^2q^2\left(A^\nu - \frac{\partial^\nu\theta}{vq}\right) + \text{higher order terms}$$

Using only the linear term for the current, put it back into the field equation for A^ν

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu_{em}$$

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = -v^2q^2\left(A^\nu - \frac{\partial^\nu\theta}{vq}\right)$$

Now recall that a gauge transformation on A^μ has the form $A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \frac{1}{q}\partial^\mu\alpha(x)$, and notice that the right hand side already has this form. As such, define

$$A'^\nu = A^\nu - \frac{\partial^\nu\theta}{vq}$$

Then the field equation becomes

$$\Box A'^\nu - \partial^\nu\partial_\mu A'^\mu = -v^2q^2A'^\nu$$

$$\boxed{(\Box + v^2q^2)A'^\nu - \partial^\nu\partial_\mu A'^\mu = 0}$$

Finally, we see that the field equation becomes a free massive vector field for a particle with mass vq. In particular, notice how the appropriate choice of gauge allowed the massless gauge field A^ν absorb the Goldstone field θ and become massive.