SU(2) Global Spontaneous Symmetry Breaking

$$\tau_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \tau_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \tau_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Consider an SU(2) doublet of bosons.

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \frac{1}{\sqrt{2}}(\phi_3 + i\phi_4) \end{bmatrix}$$

where ϕ^+ destroys ⊕ particles and creates ⊖ particles, and ϕ^0 destroys neutral particles and creates neutral antiparticles.

We shall see that this also has a U(1) symmetry, leading to a conserved quantum number y. The SU(2) symmetry is associated with the eigenvalue t_3, of the third Pauli Spin Matrix.

The Lagrangian is again

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2 \phi^\dagger \phi - \frac{\lambda}{4} |\phi^\dagger \phi|^2 \qquad (\mu^2 < 0)$$

and is invariant to both U(1) and SU(2) transformations as the α is not spatially dependent here.

$$\begin{cases} \phi \to \phi' = e^{-\frac{i}{2}\vec{\alpha}\cdot\vec{\tau}} \phi & \text{for SU(2)} \\ \phi \to \phi' = e^{-i\alpha} \phi & \text{for U(1)} \end{cases}$$

The minimum occurs at

$$\frac{\partial \mathcal{L}}{\partial(\phi^\dagger \phi)} = -\mu^2 - \frac{\lambda}{2}(\phi^\dagger \phi)_{min} = 0$$

$$(\phi^\dagger \phi)_{min} = -\frac{2\mu^2}{\lambda} \equiv \frac{v^2}{2} \qquad (\mu^2 < 0)$$

As before, we take the minimum to be the vacuum.

$$\langle 0|\phi^\dagger\phi|0\rangle = \frac{v^2}{2} = \langle 0|\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2|0\rangle$$

To obtain the particle spectrum we expand the fields ϕ about the choice of vacuum. Again, rather than a single point, we have a whole space of minima to choose from. Let,

$$\langle 0|\phi|0\rangle = \begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix}$$

Oscillations about this vacuum choice are parametrized by

$$\phi = e^{-\frac{i}{2}(\vec{\theta}(x)\cdot\vec{\tau})v}\begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+H(x)) \end{bmatrix}$$

We have here three fields for possible angular oscillations associated with the SU(2) symmetry $\vec{\theta}$, and one radial field oscillation $H(x)$. We shall see now that only the $H(x)$ field has nonzero mass, indicating that each $\vec{\theta}$ field oscillates in a direction within the minima manifold. To do this, put ϕ back into the Lagrangian and look for mass terms.

$$\partial^\mu\phi = \begin{bmatrix} 0 \\ -\frac{i}{2v}\left(\partial^\mu\vec{\theta}\cdot\vec{\tau}\right)e^{-\frac{i}{2v}\vec{\theta}\cdot\vec{\tau}}\frac{1}{\sqrt{2}}(v+H) + e^{-\frac{i}{2v}\vec{\theta}\cdot\vec{\tau}}\frac{1}{\sqrt{2}}\partial^\mu H \end{bmatrix}$$

$$= e^{-\frac{i}{2v}\vec{\Theta}\cdot\vec{\tau}} \begin{bmatrix} 0 \\ \frac{-i}{2\sqrt{2}v}(\partial^\mu\vec{\Theta}\cdot\vec{\tau})(v+H) + \frac{1}{\sqrt{2}}\partial^\mu H \end{bmatrix}$$

$$(\partial^\mu\phi)^\dagger = e^{\frac{i}{2v}\vec{\Theta}\cdot\vec{\tau}} \begin{bmatrix} 0 & , & \frac{i}{2\sqrt{2}v}(\partial^\mu\vec{\Theta}\cdot\vec{\tau})(v+H) + \frac{1}{\sqrt{2}}\partial^\mu H \end{bmatrix}$$

$$(\partial^\mu\phi)^\dagger(\partial_\mu\phi) = \left[\frac{i}{2\sqrt{2}v}(\partial^\mu\vec{\Theta}\cdot\vec{\tau})(v+H) + \frac{1}{\sqrt{2}}\partial^\mu H\right]\left[\frac{-i}{2\sqrt{2}v}(\partial_\mu\vec{\Theta}\cdot\vec{\tau})(v+H) + \frac{1}{\sqrt{2}}\partial_\mu H\right]$$

$$= \frac{1}{8v^2}(\partial^\mu\vec{\Theta}\cdot\vec{\tau})(\partial_\mu\vec{\Theta}\cdot\vec{\tau})(v+H)^2 + \frac{1}{2}(\partial^\mu H)(\partial_\mu H)$$

$$\phi^\dagger\phi = \frac{1}{2}(v+H)^2$$

$$\Rightarrow \mathcal{L} = \frac{1}{8v^2}(\partial^\mu\vec{\Theta}\cdot\vec{\tau})(\partial_\mu\vec{\Theta}\cdot\vec{\tau})(v+H)^2 + \frac{1}{2}(\partial^\mu H)(\partial_\mu H) - \frac{\mu^2}{2}(v+H)^2 - \frac{\lambda}{4}(v+H)^4$$

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Mass terms for fields have the form (const.)(field)^2.

Terms with different fields multiplied govern the interaction between the fields. Notice how since $\vec{\Theta}(x)$ only appear in an exponent in the field ϕ, it only has derivative terms in the Lagrangian. Thus, the particles associated w/ the $\vec{\Theta}$ fields are massless. Only the $H(x)$ has a mass term.