19.6 Spontaneously Broken Local $SU(2) \times U(1)$ Symmetry

Recall that we had a scalar $SU(2)$ doublet

$$\phi = \begin{vmatrix} \phi^+ \\ \phi^0 \end{vmatrix}$$

Need to show $Q = T_3 + \frac{Y}{2}$

whose Lagrangian is

$$\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) + \mu^2 \phi^\dagger \phi - \frac{\lambda}{4} (\phi^\dagger \phi)^2$$

This Lagrangian is invariant to $U(1)$ global transformations

$$\phi \rightarrow \phi' = e^{-i\alpha} \phi$$

and global $SU(2)$ transformations

$$\phi \rightarrow \phi' = e^{-\frac{i}{2}\vec{\alpha} \cdot \vec{\tau}} \phi$$

For a theory that is invariant to local transformation we must introduce three $SU(2)$ gauge fields and one $U(1)$ gauge field: $W_i^\mu(x)$ $(i=1,2,3)$; $B^\mu(x)$. Also, the derivatives must be replaced with a covariant derivative for both $U(1)$ and $SU(2)$

$$D^\mu \phi = (\partial^\mu + \underbrace{\frac{ig}{2} \vec{\tau} \cdot \vec{W}^\mu}_{\substack{SU(2) \\ \text{piece}}} + \underbrace{\frac{ig'}{2} B^\mu}_{\substack{U(1) \\ \text{piece}}}) \phi$$

Put Y quantum number in here

Kinetic terms for the new gauge fields must also be

included

$$\vec{F}^{\mu\nu} = \partial^\mu \vec{W}^\nu - \partial^\nu \vec{W}^\mu - g\vec{W}^\mu \times \vec{W}^\nu$$

$$G^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu$$

So the new full Lagrangian is

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \frac{\lambda}{4}(\phi^\dagger \phi)^2 - \frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$

We already looked at spontaneous symmetry breaking for $SU(2)\times U(1)$, so the question is how do we choose the non-zero vacuum expectation value that breaks the symmetry? For electroweak theory, we should be left with three massive gauge bosons ($W^\pm, Z$) and one massless gauge boson (photon). Being massless, the photon corresponds to some symmetry that is left unbroken. Weinberg suggested

$$\langle 0|\phi|0\rangle = \begin{bmatrix} 0 \\ \frac{\sqrt{2}\mu}{\lambda^{\frac{1}{2}}} \end{bmatrix} \equiv \begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix}$$

This choice leaves the vacuum invariant to a transformation of $U(1)$ + third component of $SU(2)$. That is,

$$\frac{1}{2}(1+\tau_3)\langle 0|\phi|0\rangle = \frac{1}{2}(1+\tau_3)\begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To consider ~~isolat~~ oscillations about the vacuum, parametrize the degrees of freedom by

$$\phi = e^{-\frac{i}{2v}\vec{\theta}(x)\cdot\vec{\tau}} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+H(x)) \end{bmatrix}$$

However, recall that the three $\vec{\theta}$ field perturbations, which would become Goldstone bosons, disappear if we make the appropriate gauge transformation. So we really just use

$$\phi = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+H(x)) \end{bmatrix}$$

$$\mathcal{L}_{free} = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{\mu^2}{2}(v+H)^2 - \frac{\lambda}{16}(v+H)^4 - \frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$

$$\begin{aligned}\mathcal{L}_{free} = {} & \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{\mu^2}{2}(v+H)^2 - \frac{\lambda}{16}(v+H)^4 \\ & - \frac{1}{4}(\partial_\mu W_{1\nu} - \partial_\nu W_{1\mu})(\partial^\mu W_1^\nu - \partial^\nu W_1^\mu) + \frac{1}{8}g^2 v^2 W_{1\mu}W_1^\mu \\ & - \frac{1}{4}(\partial_\mu W_{2\nu} - \partial_\nu W_{2\mu})(\partial^\mu W_2^\nu - \partial^\nu W_2^\mu) + \frac{1}{8}g^2 v^2 W_{2\mu}W_2^\mu \\ & - \frac{1}{4}(\partial_\mu W_{3\nu} - \partial_\nu W_{3\mu})(\partial^\mu W_3^\nu - \partial^\nu W_3^\mu) - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \\ & + \frac{1}{8}v^2(gW_{3\mu} - g'B_\mu)(gW_3^\mu - g'B^\mu)\end{aligned}$$

This is worked out in detail in "The Higgs Sector"

The second and third lines show that the W_1 and W_2 gauge fields are massive, and have the same mass $m_W = \frac{gv}{2}$. These are the $W^{\pm}$ bosons.

The last two lines show that the gauge fields W_3 and B are mixed. The key clue is to notice in the last line it is the combination $(gW_3^\mu - g' B^\mu)$ that has a mass. Introduce the linear combinations

$$\begin{cases} Z^\mu \equiv W_3^\mu \cos\theta_W - B^\mu \sin\theta_W \\ A^\mu \equiv W_3^\mu \sin\theta_W + B^\mu \cos\theta_W \end{cases}$$

where,

$$\begin{cases} \cos\theta_W = \dfrac{g}{\sqrt{g^2 + g'^2}} \\ \sin\theta_W = \dfrac{g'}{\sqrt{g^2 + g'^2}} \end{cases}$$

Or, if we invert them:

$$Z^\mu = W_3^\mu \cos\theta_W - B^\mu \sin\theta_W \qquad A^\mu = W_3^\mu \sin\theta_W + B^\mu \cos\theta_W$$

$$\frac{1}{\cos\theta} Z^\mu = W_3^\mu - B^\mu \tan\theta_W$$

$$\frac{1}{\cos\theta} Z^\mu + B^\mu \tan\theta_W = W_3^\mu$$

$$A^\mu = \left[\frac{1}{\cos\theta_W} Z^\mu + B^\mu \tan\theta_W\right] \sin\theta_W + B^\mu \cos\theta_W$$

$$A^\mu = Z^\mu \tan\theta_W + B^\mu \sin\theta_W \tan\theta_W + B^\mu \cos\theta_W$$

$$A^\mu - Z^\mu \tan\theta = B^\mu (\sin\theta \tan\theta + \cos\theta$$

$$A^\mu \cos\theta_W - Z^\mu \cos\theta_W \tan\theta_W = B^\mu (\sin^2\theta_W + \cos^2\theta_W)$$

$$\underline{A^\mu \cos\theta_W - Z^\mu \sin\theta_W = B^\mu}$$

$$\frac{1}{\cos\theta_W} Z^\mu + \left[A^\mu \cos\theta_W - Z^\mu \sin\theta_W\right] \tan\theta_W = W_3^\mu$$

$$\frac{1}{\cos\theta} Z^\mu + A^\mu \sin\theta - Z^\mu \frac{\sin^2\theta_W}{\cos\theta_W} = W_3^\mu$$

$$Z^\mu - Z^\mu \sin^2\theta_W + A^\mu \sin\theta_W \cos\theta_W = W_3 \cos\theta_W$$

$$Z^\mu (1 - \sin^2\theta_W) + A^\mu \sin\theta_W \cos\theta_W = W_3^\mu \cos\theta_W$$

$$Z^\mu \cos^2\theta_W + A^\mu \sin\theta_W \cos\theta_W = W_3^\mu \cos\theta_W$$

$$\underline{Z^\mu \cos\theta_W + A^\mu \sin\theta_W = W_3^\mu}$$

$$\Longrightarrow \begin{cases} B^\mu = A^\mu \cos\theta_W - Z^\mu \sin\theta_W \\ W_3^\mu = A^\mu \sin\theta_W + Z^\mu \cos\theta_W \end{cases}$$

Using this, we can write the last two lines of the Lagrangian in terms of A^μ and Z^μ, instead of B^μ and W_3^μ.

$$-\frac{1}{4}(\partial_\mu W_{3\nu} - \partial_\nu W_{3\mu})(\partial^\mu W_3^\nu - \partial^\nu W_3^\mu) - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + \frac{1}{8} v^2 (g W_{3\mu} - g' B_\mu)(g W_3^\mu - g' B^\mu)$$

$$\Rightarrow -\frac{1}{4}\left(\partial_\mu(Z_\nu\cos\theta_W + A_\nu\sin\theta_W) - \partial_\nu(Z_\mu\cos\theta_W + A_\mu\sin\theta_W)\right)$$
$$\cdot\left(\partial^\mu(Z^\nu\cos\theta_W + A^\nu\sin\theta_W) - \partial^\nu(Z^\mu\cos\theta_W + A^\mu\sin\theta_W)\right)$$

$$-\frac{1}{4}\left(\partial_\mu(A_\nu\cos\theta_W - Z_\nu\sin\theta_W) - \partial_\nu(A_\mu\cos\theta_W - Z_\mu\sin\theta_W)\right)$$
$$\cdot\left(\partial^\mu(A^\nu\cos\theta_W - Z^\nu\sin\theta_W) - \partial^\nu(A^\mu\cos\theta_W - Z^\mu\sin\theta_W)\right)$$

$$+\frac{1}{8}v^2\left(g(Z_\mu\cos\theta_W + A_\mu\sin\theta_W) - g'(A_\mu\cos\theta_W - Z_\mu\sin\theta_W)\right)$$
$$\cdot\left(g(Z^\mu\cos\theta_W + A^\mu\sin\theta_W) - g'(A^\mu\cos\theta_W - Z^\mu\sin\theta_W)\right)$$

$$\Rightarrow -\frac{1}{4}\left((\partial_\mu Z_\nu - \partial_\nu Z_\mu)\cos\theta_W + (\partial_\mu A_\nu - \partial_\nu A_\mu)\sin\theta_W\right)$$
$$\cdot\left((\partial^\mu Z^\nu - \partial^\nu Z^\mu)\cos\theta_W + (\partial^\mu A^\nu - \partial^\nu A^\mu)\sin\theta_W\right)$$

$$-\frac{1}{4}\left((\partial_\mu A_\nu - \partial_\nu A_\mu)\cos\theta_W - (\partial_\mu Z_\nu - \partial_\nu Z_\mu)\sin\theta_W\right)$$
$$\cdot\left((\partial^\mu A^\nu - \partial^\nu A^\mu)\cos\theta_W - (\partial^\mu Z^\nu - \partial^\nu Z^\mu)\sin\theta_W\right)$$

$$+\frac{1}{8}v^2\left(Z_\mu(g\cos\theta_W + g'\sin\theta_W) + A_\mu(g\sin\theta_W - g'\cos\theta_W)\right)$$
$$\cdot\left(Z^\mu(g\cos\theta_W + g'\sin\theta_W) + A^\mu(g\sin\theta_W - g'\cos\theta_W)\right)$$

Define $A_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$; $Z_{\mu\nu} \equiv \partial_\mu Z_\nu - \partial_\nu Z_\mu$.

$$\Rightarrow -\frac{1}{4}\left(Z_{\mu\nu}\cos\theta_W + \mathcal{A}_{\mu\nu}\sin\theta_W\right)\left(Z^{\mu\nu}\cos\theta_W + \mathcal{A}^{\mu\nu}\sin\theta_W\right)$$

$$-\frac{1}{4}\left(\mathcal{A}_{\mu\nu}\cos\theta_W - Z_{\mu\nu}\sin\theta_W\right)\left(\mathcal{A}^{\mu\nu}\cos\theta_W - Z^{\mu\nu}\sin\theta_W\right)$$

$$+\frac{1}{8}v^2\left(Z_\mu\left(\frac{g^2+g'^2}{\sqrt{g^2+g'^2}}\right) + A_\mu\left(\frac{gg'-g'g}{\sqrt{g^2-g'^2}}\right)\right)$$

$$\cdot\left(Z^\mu\left(\frac{g^2+g'^2}{\sqrt{g^2+g'^2}}\right) + A^\mu\left(\frac{gg'-g'g}{\sqrt{g^2+g'^2}}\right)\right)$$

$$\Rightarrow -\frac{1}{4}\left(Z_{\mu\nu}Z^{\mu\nu}\cos^2\theta_W + Z_{\mu\nu}\mathcal{A}^{\mu\nu}\sin\theta_W\cos\theta_W + \mathcal{A}_{\mu\nu}Z^{\mu\nu}\sin\theta_W\cos\theta_W + \mathcal{A}_{\mu\nu}\mathcal{A}^{\mu\nu}\sin^2\theta_W\right.$$

$$+\mathcal{A}_{\mu\nu}\mathcal{A}^{\mu\nu}\cos^2\theta_W - \mathcal{A}_{\mu\nu}Z^{\mu\nu}\sin\theta_W\cos\theta_W - Z_{\mu\nu}\mathcal{A}^{\mu\nu}\sin\theta_W\cos\theta_W + Z_{\mu\nu}Z^{\mu\nu}\sin^2\theta_W$$

$$+\frac{1}{8}v^2 Z_\mu Z^\mu\left(g^2+g'^2\right)$$

$$\boxed{\Rightarrow -\frac{1}{4}\left(Z_{\mu\nu}Z^{\mu\nu} + \mathcal{A}_{\mu\nu}\mathcal{A}^{\mu\nu}\right) + \frac{1}{8}v^2 Z_\mu Z^\mu\left(g^2+g'^2\right)}$$

Thus, we have unmixed the two fields. They become the Z^0 boson and the photon.

$$\begin{cases} m_Z = \frac{1}{2}v^2\sqrt{g^2+g'^2} = \dfrac{m_W}{\cos\theta_W} \\ m_A = 0 \end{cases}$$