$SU(2) \times U(1)$ Gauge Invariant Lagrangian (massless)

Let's look at $SU(2) \times U(1)$ gauge invariance for the first generation of leptons; the consequences are identical for the muon and tau generations. The Higgs mechanism is NOT included here so are particles are still massless. The Lagrangian density is split between left-handed doublets $l_L = \begin{bmatrix} \nu_L \\ e_L \end{bmatrix}$ and right handed singlets $l_R = e_R$, where for some field ψ

$$\begin{cases} \psi_L = \left(\frac{1-\gamma_5}{2}\right)\psi \\ \psi_R = \left(\frac{1+\gamma_5}{2}\right)\psi \end{cases}$$

Do this for a general ~~lepton~~ doublet to cover leptons and quarks.

Hence,

$$\mathcal{L} = \bar{l}\, i \not{D}\, l$$

$$\mathcal{L} = \underbrace{\bar{l}_L\, i \not{D}\, l_L}_{\mathcal{L}_L} + \underbrace{\bar{l}_R\, i \not{D}\, l_R}_{\mathcal{L}_R}$$

Point out the inclusion of quantum number "y", it significance to U(1), and various values for quarks and leptons.

where
$$\begin{cases} D_L^\rho = \partial^\rho + \frac{ig}{2}\vec{\tau}\cdot\vec{W}^\rho + \frac{ig'y}{2}B^\rho \\ D_R^\rho = \partial^\rho + \frac{ig'y}{2}B^\rho \end{cases}$$

I. Look at the $\mathcal{L}_R$ terms

$$\mathcal{L}_R = \bar{l}_R\, i \not{D}_R\, l_R$$
$$= \bar{l}_R\, i\gamma_\rho\left(\partial^\rho + \frac{ig'y}{2}B^\rho\right) l_R$$

$$\mathcal{L}_R = i\bar{e}_R \gamma_\rho \partial^\rho e_R - \frac{g'y}{2} \bar{e}_R \gamma_\rho B^\rho e_R$$

$$= i e_R^\dagger \gamma_0 \gamma_\rho \partial^\rho e_R - \frac{g'y}{2} e_R^\dagger \gamma_0 \gamma_\rho B^\rho e_R$$

$$= i e^\dagger \left(\frac{1+\gamma_5}{2}\right) \gamma_0 \gamma_\rho \left(\frac{1+\gamma_5}{2}\right) \partial^\rho e - \frac{g'y}{2} e^\dagger \left(\frac{1+\gamma_5}{2}\right) \gamma_0 \gamma_\rho B^\rho \left(\frac{1+\gamma_5}{2}\right) e$$

Use the fact that γ_5 anticommutes with the other γ_μ's, so $\{\gamma_5, \gamma_\mu\} = 0 \Rightarrow \left(\frac{1+\gamma_5}{2}\right)\gamma_\mu = \left(\frac{\gamma_\mu + \gamma_5\gamma_\mu}{2}\right) = \left(\frac{\gamma_\mu - \gamma_\mu\gamma_5}{2}\right) = \gamma_\mu\left(\frac{1-\gamma_5}{2}\right)$.

Also, note that after $\left(\frac{1+\gamma_5}{2}\right)$ commutes past $\gamma_0\gamma_\rho$,

$\left(\frac{1+\gamma_5}{2}\right)\left(\frac{1+\gamma_5}{2}\right) = \left(\frac{1+\gamma_5}{2}\right)$.

$$\Rightarrow \mathcal{L}_R = i\bar{e}\gamma_\rho \left(\frac{1+\gamma_5}{2}\right)\partial^\rho e - \frac{g'y}{2} \bar{e}\gamma_\rho B^\rho \left(\frac{1+\gamma_5}{2}\right) e$$

We'll come back to these terms later.

II. Look at the $\mathcal{L}_L$ terms

$$\mathcal{L}_L = \bar{l}_L i \not{D} l_L$$

Before proceeding, be careful to note that $\psi_L = \left(\frac{1-\gamma_5}{2}\right)\psi$ for the doublet means

$$l_L = \begin{bmatrix} \nu_L \\ e_L \end{bmatrix}$$

and then $\bar{e}_L = e_L^\dagger \gamma_0 = \left(\left(\frac{1-\gamma_5}{2}\right)e\right)^\dagger \gamma_0 = e^\dagger \left(\frac{1-\gamma_5}{2}\right)^\dagger \gamma_0 = e^\dagger \left(\frac{1-\gamma_5}{2}\right)\gamma_0$,

and the same goes for $\bar{\nu}_L$. Further,

$\bar{l}_L = \overline{\begin{bmatrix} \nu_L \\ e_L \end{bmatrix}} = \begin{bmatrix} \bar{\nu}_L & \bar{e}_L \end{bmatrix}$. Now we're ready for the matrix algebra.

$$\mathcal{L}_L = \begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix} i\gamma_\rho D^\rho \begin{bmatrix}\nu_L \\ e_L\end{bmatrix}$$

$$= \begin{bmatrix}\bar{\nu}_L & e_L\end{bmatrix} i\gamma_\rho \left(\partial^\rho + \frac{ig}{2}\vec{\tau}\cdot\vec{W}^\rho + \frac{ig'y}{2}B^\rho\right) \begin{bmatrix}\nu_L \\ e_L\end{bmatrix}$$

$$= \underbrace{i\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix}\gamma_\rho\partial^\rho \begin{bmatrix}\nu_L \\ e_L\end{bmatrix}}_{(\mathrm{II.A})} - \underbrace{\frac{g}{2}\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix}\gamma_\rho\vec{\tau}\cdot\vec{W}^\rho \begin{bmatrix}\nu_L \\ e_L\end{bmatrix}}_{(\mathrm{II.B})} - \underbrace{\frac{g'y}{2}\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix}\gamma_\rho B^\rho \begin{bmatrix}\nu_L \\ e_L\end{bmatrix}}_{(\mathrm{II.C})}$$

II.A. The Derivative Terms

$$i\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix}\gamma_\rho\partial^\rho \begin{bmatrix}\nu_L \\ e_L\end{bmatrix} = i\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix} \begin{bmatrix}\not{\partial}\nu_L \\ \not{\partial}e_L\end{bmatrix}$$

$$= i\bar{\nu}_L\not{\partial}\nu_L + i\bar{e}_L\not{\partial}e_L$$

$$= i\nu_L^\dagger\gamma_0\gamma_\rho\partial^\rho\nu_L + ie_L^\dagger\gamma_0\gamma_\rho\partial^\rho e_L$$

$$= i\nu^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho\nu + ie^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho e$$

$$= i\nu^\dagger\gamma_0\left(\frac{1+\gamma_5}{2}\right)\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho\nu + ie^\dagger\gamma_0\left(\frac{1+\gamma_5}{2}\right)\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho e$$

$$= i\nu^\dagger\gamma_0\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)^2\partial^\rho\nu + ie^\dagger\gamma_0\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)^2\partial^\rho e$$

$$= i\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho\nu + i\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho e$$

II.B The $W, W^\dagger, W_3$ Terms

$$\frac{g}{2}\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix}\gamma_\rho\vec{\tau}\cdot\vec{W}^\rho \begin{bmatrix}\nu_L \\ e_L\end{bmatrix}$$

The key here is to re-express

$$\frac{1}{2}\vec{\tau}\cdot\vec{W}^\mu = \frac{1}{2}\left[\tau_1 W_1^\mu + \tau_2 W_2^\mu + \tau_3 W_3^\mu\right]$$

$$= \frac{1}{\sqrt{2}}\left[\tau_+\left(\frac{W_1^\mu - iW_2^\mu}{\sqrt{2}}\right) + \tau_-\left(\frac{W_1^\mu + iW_2^\mu}{\sqrt{2}}\right)\right] + \frac{\tau_3}{2}W_3^\mu$$

$$= \frac{1}{\sqrt{2}}\left[\tau_+ W^\mu + \tau_- W^{\dagger\mu}\right] + \frac{\tau_3}{2}W_3^\mu$$

where

$$\begin{cases} \tau_+ = \frac{1}{2}(\tau_1 + i\tau_2) = \frac{1}{2}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \tau_- = \frac{1}{2}(\tau_1 - i\tau_2) = \frac{1}{2}\left\{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ W^\mu = \frac{W_1^\mu - iW_2^\mu}{\sqrt{2}} \\ W^{\dagger\mu} = \frac{W_1^\mu + iW_2^\mu}{\sqrt{2}} \end{cases}$$

The reason for this is to do the following:

$$\frac{1}{\sqrt{2}}\tau_+ W^\mu \begin{bmatrix} \nu_L \\ e_L \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & W^\mu \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \nu_L \\ e_L \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} W^\mu e_L \\ 0 \end{bmatrix} = \frac{W^\mu}{\sqrt{2}}\begin{bmatrix} e_L \\ 0 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\tau_- W^{\dagger\mu} \begin{bmatrix} \nu_L \\ e_L \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ W^{\dagger\mu} & 0 \end{bmatrix}\begin{bmatrix} \nu_L \\ e_L \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ W^{\dagger\mu}\nu_L \end{bmatrix} = \frac{W^{\dagger\mu}}{\sqrt{2}}\begin{bmatrix} 0 \\ \nu_L \end{bmatrix}$$

Notice how the e_L and ν_L fields switch positions in the vector. This is what will subsequently allow interactions between these fields via the gauge bosons $W^\mu, W^{\dagger\mu}$.

Lastly,

$$\frac{1}{2}\tau_3 W_3^\mu \begin{bmatrix} \nu_L \\ e_L \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} W_3^\mu \nu_L \\ W_3^\mu e_L \end{bmatrix} = \frac{1}{2}\begin{bmatrix} W_3^\mu \nu_L \\ -W_3^\mu e_L \end{bmatrix} = \frac{W_3^\mu}{2}\begin{bmatrix} \nu_L \\ -e_L \end{bmatrix}$$

Now we're ready to get back to the term from the Lagrangian.

$$\frac{g}{2}\begin{bmatrix} \bar{\nu}_L & \bar{e}_L \end{bmatrix}\gamma_\rho \vec{\tau}\cdot\vec{W}^\rho \begin{bmatrix} \nu_L \\ e_L \end{bmatrix}$$

$$= g\begin{bmatrix} \bar{\nu}_L & \bar{e}_L \end{bmatrix}\gamma_\rho\left(\frac{1}{2}\tau_+(W_1^\rho - iW_2^\rho) + \frac{1}{2}\tau_-(W_1^\rho + iW_2^\rho) + \frac{1}{2}\tau_3 W_3^\rho\right)\begin{bmatrix} \nu_L \\ e_L \end{bmatrix}$$

$$= g\begin{bmatrix} \bar{\nu}_L & \bar{e}_L \end{bmatrix}\gamma_\rho\left(\frac{1}{\sqrt{2}}\tau_+ W^\rho + \frac{1}{\sqrt{2}}\tau_- W^{\dagger\rho} + \frac{1}{2}\tau_3 W_3^\rho\right)\begin{bmatrix} \nu_L \\ e_L \end{bmatrix}$$

$$= g\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix}\gamma_\rho\left(\frac{1}{\sqrt{2}}W^\rho\begin{bmatrix}e_L\\0\end{bmatrix}+\frac{1}{\sqrt{2}}W^{\dagger\rho}\begin{bmatrix}0\\\nu_L\end{bmatrix}+\frac{1}{2}W_3^\rho\begin{bmatrix}\nu_L\\-e_L\end{bmatrix}\right)$$

$$= \frac{g}{\sqrt{2}}\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix}\begin{bmatrix}\not{W}^\rho e_L\\0\end{bmatrix}+\frac{g}{\sqrt{2}}\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix}\begin{bmatrix}0\\\not{W}^{\dagger\rho}\nu_L\end{bmatrix}+\frac{g}{2}\begin{bmatrix}\bar{\nu}_L & \bar{e}_L\end{bmatrix}\begin{bmatrix}\not{W}_3^\rho\nu_L\\-\not{W}_3^\rho e_L\end{bmatrix}$$

$$= \frac{g}{\sqrt{2}}\bar{\nu}_L\not{W}^\rho e_L+\frac{g}{\sqrt{2}}\bar{e}_L\not{W}^{\dagger\rho}\nu_L+\frac{g}{2}\bar{\nu}_L\not{W}_3^\rho\nu_L-\frac{g}{2}\bar{e}_L\not{W}_3^\rho e_L$$

$$= \frac{g}{\sqrt{2}}\nu_L^\dagger\gamma_0\gamma_\rho W^\rho e_L+\frac{g}{\sqrt{2}}e_L^\dagger\gamma_0\gamma_\rho W^{\dagger\rho}\nu_L+\frac{g}{2}\nu_L^\dagger\gamma_0\gamma_\rho W_3^\rho\nu_L-\frac{g}{2}e_L^\dagger\gamma_0 W_3^\rho e_L$$

$$= \frac{g}{\sqrt{2}}\nu^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho W^\rho\left(\frac{1-\gamma_5}{2}\right)e+\frac{g}{\sqrt{2}}e^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho W^{\dagger\rho}\left(\frac{1-\gamma_5}{2}\right)\nu$$

$$+\frac{g}{2}\nu^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho W_3^\rho\left(\frac{1-\gamma_5}{2}\right)\nu-\frac{g}{2}e^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho W_3^\rho\left(\frac{1-\gamma_5}{2}\right)e$$

$$= \frac{g}{\sqrt{2}}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)eW^\rho+\frac{g}{\sqrt{2}}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu W^{\dagger\rho}$$

$$+\frac{g}{2}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu W_3^\rho-\frac{g}{2}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)eW_3^\rho$$

The 1st term governs interactions between the electron, W^-, and antineutrino; the 2nd term governs interactions between the positron, W^+, and neutrino. The 3rd and 4th terms will have to mix with the B^ρ terms to govern how electrons (and positrons) and neutrinos interact via the photon and Z^0! We also have now the vertex factor in the invariant amplitude iM for e, ν, W interactions

$$-\frac{ig}{\sqrt{2}}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)$$

II.C. The B^ρ Field

$$\frac{g'y}{2}\left[\bar{\nu}_L \;\; \bar{e}_L\right]\gamma_\rho B^\rho \begin{bmatrix}\nu_L \\ e_L\end{bmatrix}$$

$$= \frac{g'y}{2}\left[\bar{\nu}_L \;\; \bar{e}_L\right]\begin{bmatrix}\not{B}\nu_L \\ \not{B}e_L\end{bmatrix}$$

$$= \frac{g'y}{2}\left[\bar{\nu}_L \not{B}\nu_L + \bar{e}_L \not{B} e_L\right]$$

$$= \frac{g'y}{2}\left[\nu_L^\dagger \gamma_0 \gamma_\rho B^\rho \nu_L + e_L^\dagger \gamma_0 \gamma_\rho B^\rho e_L\right]$$

$$= \frac{g'y}{2}\left[\nu^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho B^\rho\left(\frac{1-\gamma_5}{2}\right)\nu + e^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho B^\rho\left(\frac{1-\gamma_5}{2}\right)e\right]$$

$$= \frac{g'y}{2}\left[\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu B^\rho + \bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e B^\rho\right]$$

III. Find the Z^0 and Photon Interactions

The next task is to mix these terms with the W_3^μ terms from before to yield the photon and Z^0-boson interactions with leptons and their neutrinos. Recall I found in "SU(2)×U(1) Local Spontaneously Broken Symmetry" that the mixing is given by

$$\begin{bmatrix} Z^\mu \\ A^\mu \end{bmatrix} = \begin{bmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{bmatrix}\begin{bmatrix} W_3^\mu \\ B^\mu \end{bmatrix}$$

where
$$\begin{cases} \cos\theta_W = \dfrac{g}{\sqrt{g^2+g'^2}} \\[2ex] \sin\theta_W = \dfrac{g'}{\sqrt{g^2+g'^2}} \end{cases}$$

Or inverted as

$$\begin{bmatrix} W_3^\mu \\ B^\mu \end{bmatrix} = \begin{bmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{bmatrix} \begin{bmatrix} Z^\mu \\ A^\mu \end{bmatrix}$$

Gather all the W_3^ρ and B^ρ terms from the Lagrangian density.

$$-\frac{g}{2}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu W_3^\rho + \frac{g}{2}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e W_3^\rho - \frac{g'y}{2}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu B^\rho$$

$$-\frac{g'y}{2}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e B^\rho - \frac{g'y}{2}\bar{e}\gamma_\rho B^\rho\left(\frac{1+\gamma_5}{2}\right)e$$

$$= -\frac{g}{2}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu\left(Z^\rho\cos\theta_W + A^\rho\sin\theta_W\right)$$

$$+\frac{g}{2}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e\left(Z^\rho\cos\theta_W + A^\rho\sin\theta_W\right)$$

$$-\frac{g'y}{2}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu\left(-Z^\rho\sin\theta_W + A^\rho\cos\theta_W\right)$$

$$-\frac{g'y}{2}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e\left(-Z^\rho\sin\theta_W + A^\rho\cos\theta_W\right)$$

$$-\frac{g'y}{2}\bar{e}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)e\left(-Z^\rho\sin\theta_W + A^\rho\cos\theta_W\right)$$

$$= -\frac{g}{2}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu\left(Z^\rho\cos\theta_W + A^\rho\sin\theta_W\right)$$

$$+\frac{g}{2}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e\left(Z^\rho\cos\theta_W + A^\rho\sin\theta_W\right)$$

$$+\frac{g'}{2}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu\left(-Z^\rho\sin\theta_W + A^\rho\cos\theta_W\right)$$

$$+\frac{g'}{2}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e\left(-Z^\rho\sin\theta_W + A^\rho\cos\theta_W\right)$$

$$+g'\bar{e}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)e\left(-Z^\rho\sin\theta_W + A^\rho\cos\theta_W\right)$$

where $y=-1$ for left-handed doublets and $y=-2$ for right-handed singlets (AH p.340, table 22.1).

$$= \frac{1}{2}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu Z^\rho\left(-g\cos\theta_W - g'\sin\theta_W\right)$$

$$+ \frac{1}{2}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu A^\rho\left(-g\sin\theta_W + g'\cos\theta_W\right)$$

$$+ \frac{1}{2}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e Z^\rho\left(g\cos\theta_W - g'\sin\theta_W\right)$$

$$+ \frac{1}{2}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e A^\rho\left(g\sin\theta_W + g'\cos\theta_W\right)$$

$$- \bar{e}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)e Z^\rho\left(g'\sin\theta_W\right) + \bar{e}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)e A^\rho\left(g'\cos\theta_W\right)$$

Use $g' = g\dfrac{\sin\theta_W}{\cos\theta_W}$ in all terms.

$$= \frac{1}{2}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu Z^\rho\left(-g\frac{\cos^2\theta_W}{\cos\theta_W} - g\frac{\sin^2\theta_W}{\cos\theta_W}\right)$$

$$\cancel{+ \frac{1}{2}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu A^\rho\left(-g\sin\theta_W + g\frac{\sin\theta_W}{\cos\theta_W}\cdot\cos\theta_W\right)} \to 0$$

$$+ \frac{1}{2}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e Z^\rho\left(g\frac{\cos^2\theta_W}{\cos\theta_W} - g\frac{\sin^2\theta_W}{\cos\theta_W}\right)$$

$$+ \frac{1}{2}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e A^\rho\left(g\sin\theta_W + g\frac{\sin\theta_W}{\cancel{\cos\theta_W}}\cancel{\cos\theta_W}\right)$$

$$- \bar{e}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)e Z^\rho\left(g\frac{\sin^2\theta_W}{\cos\theta_W}\right) + \bar{e}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)e A^\rho\left(g\frac{\sin\theta_W}{\cancel{\cos\theta_W}}\cdot\cancel{\cos\theta_W}\right)$$

The disappearing term signifies that neutrinos do not couple to photons

$$= -\frac{g}{2\cos\theta_W}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu Z^\rho + \frac{g}{2\cos\theta_W}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e Z^\rho\left(1-2\sin^2\theta_W\right)$$

$$+ g\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e A^\rho\left(\sin\theta_W\right) - \frac{g}{2\cos\theta_W}\bar{e}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)e Z^\rho\left(2\sin^2\theta_W\right)$$

$$+ g\bar{e}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)e A^\rho\left(\sin\theta_W\right)$$

Since the photon interactions must be consistent with Q.E.D. there are no left and right handed photons, i.e. no $\left(\frac{1\pm\gamma_5}{2}\right)$ factors. Indeed, $\left(\frac{1-\gamma_5}{2}\right)+\left(\frac{1+\gamma_5}{2}\right)=1$ yields just this term for the [redacted] two terms with A^ρ.

$$= +\frac{g}{2\cos\theta_W} Z^\rho \left[\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu + \bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e\left(1-2\sin^2\theta_W\right) - \bar{e}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)e\left(2\sin^2\theta_W\right)\right]$$

$$+ \bar{e}\gamma_\rho e A^\rho e_0$$

where $e_0 \equiv g\sin\theta_W$ is the electric charge.

Consequences:

1. The 1st term governs interactions between neutrinos and the Z^0-boson. The vertex factor is

$$\frac{g}{2\cos\theta_W}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)$$

2. The 2nd term governs interactions between left-handed electrons (and positrons) and the Z^0 boson. The vertex factor is

$$\frac{g}{2\cos\theta_W}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\left(1-2\sin^2\theta_W\right)$$

3. The 3rd term is for right-handed electrons and Z^0

$$\frac{g}{2\cos\theta_W}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)\left(2\sin^2\theta_W\right)$$

4. The 4th term governs interactions between electrons and the photon. The vertex factor is

$$\gamma_\rho e_0$$

5. The full Lagrangian density is

$$\mathcal{L} = i\bar{e}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)\partial^\rho e + i\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho e + i\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho \nu$$

$$- \frac{g}{\sqrt{2}}\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)eW^\rho - \frac{g}{\sqrt{2}}\bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu W^{\dagger\rho}$$

$$+ \frac{g}{2\cos\theta_W} Z^\rho\left[-\bar{\nu}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\nu + \bar{e}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)e\left(1-2\sin^2\theta_W\right) - \bar{e}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)e\left(2\sin^2\theta_W\right)\right]$$

$$+ \bar{e}\gamma_\rho e A^\rho e_0$$

This is a repeat of the previous calculation for a quark doublet. The only major differences is a change in the values of the hypercharge and the presence of right-handed singlets for both particles. Use the quark doublet

$$q = \begin{bmatrix} u \\ d \end{bmatrix}$$

and recall

$$\begin{cases} \psi_L = \left(\frac{1-\gamma_5}{2}\right)\psi \\ \psi_R = \left(\frac{1+\gamma_5}{2}\right)\psi \end{cases}$$

So, start again with the Lagrangian (Dirac Lagrangian, instead of Klein-Gordon now)

$$\mathcal{L} = \bar{q}\, i\not{D}\, q$$

$$\mathcal{L} = \bar{q}_L\, i\not{D}\, q_L + \bar{u}_R\, i\not{D}\, u_R + \bar{d}_R\, i\not{D}\, d_R$$

where
$$\begin{cases} D_L^\rho = \partial^\rho + \frac{ig}{2}\vec{\tau}\cdot\vec{W}^\rho + \frac{ig'y}{2}B^\rho \\ D_R^\rho = \partial^\rho + \frac{ig'y}{2}B^\rho \end{cases}$$

I. Look at the $\mathcal{L}_R$ terms

$$\mathcal{L}_R = \bar{u}_R\, i\not{D}\, u_R + \bar{d}_R\, i\not{D}\, d_R$$

$$= \bar{u}_R\, i\gamma_\rho\left(\partial^\rho + \frac{ig'y}{2}B^\rho\right)u_R + \bar{d}_R\, i\gamma_\rho\left(\partial^\rho + \frac{ig'y}{2}B^\rho\right)d_R$$

$$= i\bar{u}_R\gamma_\rho\partial^\rho u_R - \frac{g'y}{2}\bar{u}_R\gamma_\rho B^\rho u_R + i\bar{d}_R\gamma_\rho\partial^\rho d_R - \frac{g'y}{2}\bar{d}_R\gamma_\rho B^\rho d_R$$

$$\mathcal{L}_R = i u_R^\dagger \gamma_0 \gamma_\rho \partial^\rho u_R - \frac{g'y}{2} u_R^\dagger \gamma_0 \gamma_\rho B^\rho u_R + i d_R^\dagger \gamma_0 \gamma_\rho \partial^\rho d_R - \frac{g'y}{2} d_R^\dagger \gamma_0 \gamma_\rho B^\rho d_R$$

$$= i u^\dagger \left(\frac{1+\gamma_5}{2}\right) \gamma_0 \gamma_\rho \left(\frac{1+\gamma_5}{2}\right) \partial^\rho u - \frac{g'y}{2} u^\dagger \left(\frac{1+\gamma_5}{2}\right) \gamma_0 \gamma_\rho B^\rho \left(\frac{1+\gamma_5}{2}\right) u$$

$$+ i d^\dagger \left(\frac{1+\gamma_5}{2}\right) \gamma_0 \gamma_\rho \left(\frac{1+\gamma_5}{2}\right) \partial^\rho d - \frac{g'y}{2} d^\dagger \left(\frac{1+\gamma_5}{2}\right) \gamma_0 \gamma_\rho B^\rho \left(\frac{1+\gamma_5}{2}\right) d$$

Use the fact that γ_5 anticommutes with the other γ_μ's,

so $\{\gamma_5, \gamma_\mu\} = 0 \Rightarrow \left(\frac{1+\gamma_5}{2}\right)\gamma_\mu = \left(\frac{\gamma_\mu + \gamma_5\gamma_\mu}{2}\right) = \left(\frac{\gamma_\mu + (-\gamma_\mu\gamma_5)}{2}\right) = \gamma_\mu\left(\frac{1-\gamma_5}{2}\right)$.

Also, note that after $\left(\frac{1+\gamma_5}{2}\right)$ commutes past $\gamma_0\gamma_\rho$,

$\left(\frac{1+\gamma_5}{2}\right)\left(\frac{1+\gamma_5}{2}\right) = \left(\frac{1+\gamma_5}{2}\right)$.

$$\Rightarrow \mathcal{L}_R = i\bar{u}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)\partial^\rho u - \frac{g'y}{2}\bar{u}\gamma_\rho B^\rho\left(\frac{1+\gamma_5}{2}\right)u$$

$$+ i\bar{d}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)\partial^\rho d - \frac{g'y}{2}\bar{d}\gamma_\rho B^\rho\left(\frac{1+\gamma_5}{2}\right)d$$

We'll come back to these terms later.

II. Look at the $\mathcal{L}_L$ terms

$\mathcal{L}_L = \bar{q}_L i \not{D} q_L$

As before, note that $\bar{u}_L = u^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0$ and also that

$$\bar{q}_L = \overline{\begin{bmatrix} u_L \\ d_L \end{bmatrix}} = \begin{bmatrix} \bar{u}_L & \bar{d}_L \end{bmatrix}$$

$$\mathcal{L}_L = \begin{bmatrix}\bar{u}_L & \bar{d}_L\end{bmatrix} i\gamma_\rho D^\rho \begin{bmatrix}u_L \\ d_L\end{bmatrix}$$

$$= \begin{bmatrix}\bar{u}_L & \bar{d}_L\end{bmatrix} i\gamma_\rho \left(\partial^\rho + \frac{ig}{2}\vec{\tau}\cdot\vec{W}^\rho + \frac{ig'y}{2}B^\rho\right) \begin{bmatrix}u_L \\ d_L\end{bmatrix}$$

$$= \underbrace{i\begin{bmatrix}\bar{u}_L & \bar{d}_L\end{bmatrix}\gamma_\rho \partial^\rho \begin{bmatrix}u_L \\ d_L\end{bmatrix}}_{(\mathrm{II.A})} \underbrace{- \frac{g}{2}\begin{bmatrix}\bar{u}_L & \bar{d}_L\end{bmatrix}\gamma_\rho \vec{\tau}\cdot\vec{W}^\rho \begin{bmatrix}u_L \\ d_L\end{bmatrix}}_{(\mathrm{II.B})} \underbrace{- \begin{bmatrix}\bar{u}_L & \bar{d}_L\end{bmatrix}\frac{g'y}{2}\gamma_\rho B^\rho \begin{bmatrix}u_L \\ d_L\end{bmatrix}}_{(\mathrm{II.C})}$$

II.A The Derivative Terms

$$i\begin{bmatrix}\bar{u}_L & \bar{d}_L\end{bmatrix}\gamma_\rho \partial^\rho \begin{bmatrix}u_L \\ d_L\end{bmatrix} = i\begin{bmatrix}\bar{u}_L & \bar{d}_L\end{bmatrix}\begin{bmatrix}\not{\partial} u_L \\ \not{\partial} d_L\end{bmatrix}$$

$$= i\bar{u}_L \not{\partial} u_L + i\bar{d}_L \not{\partial} d_L$$

$$= iu^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho u + id^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho d$$

$$= iu^\dagger\gamma_0\left(\frac{1+\gamma_5}{2}\right)\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho u + id^\dagger\gamma_0\left(\frac{1+\gamma_5}{2}\right)\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho d$$

$$= iu^\dagger\gamma_0\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)^2\partial^\rho u + id^\dagger\gamma_0\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)^2\partial^\rho d$$

$$= iu^\dagger\gamma_0\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho u + id^\dagger\gamma_0\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho d$$

$$= i\bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho u + i\bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho d$$

II.B The $W, W^\dagger, W_3$ Terms

$$\frac{g}{2}\begin{bmatrix}\bar{u}_L & \bar{d}_L\end{bmatrix}\gamma_\rho \vec{\tau}\cdot\vec{W}^\rho \begin{bmatrix}u_L \\ d_L\end{bmatrix}$$

The key here is to express

$$\frac{1}{2}\vec{\tau}\cdot\vec{W}^{\mu} = \frac{1}{2}\left[\tau_1 W_1^{\mu} + \tau_2 W_2^{\mu} + \tau_3 W_3^{\mu}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\tau_+\left(\frac{W_1^{\mu} - iW_2^{\mu}}{\sqrt{2}}\right) + \tau_-\left(\frac{W_1^{\mu} + iW_2^{\mu}}{\sqrt{2}}\right)\right] + \frac{1}{2}\tau_3 W_3^{\mu}$$

$$= \frac{1}{\sqrt{2}}\left[\tau_+ W^{\mu} + \tau_- W^{\dagger\mu}\right] + \frac{\tau_3}{2} W_3^{\mu}$$

where

$$\begin{cases} \tau_+ = \frac{1}{2}(\tau_1 + i\tau_2) = \frac{1}{2}\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right] = \frac{1}{2}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \tau_- = \frac{1}{2}(\tau_1 - i\tau_2) = \frac{1}{2}\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right] = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ W^{\mu} = \frac{W_1^{\mu} - iW_2^{\mu}}{\sqrt{2}} \\ W^{\dagger\mu} = \frac{W_1^{\mu} + iW_2^{\mu}}{\sqrt{2}} \end{cases}$$

The reason for this is to do the following:

$$\frac{1}{\sqrt{2}}\tau_+ W^{\mu}\begin{bmatrix} u_L \\ d_L \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & W^{\mu} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_L \\ d_L \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} W^{\mu} d_L \\ 0 \end{bmatrix} = \frac{W^{\mu}}{\sqrt{2}}\begin{bmatrix} d_L \\ 0 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}\tau_- W^{\dagger\mu}\begin{bmatrix} u_L \\ d_L \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 \\ W^{\dagger\mu} & 0 \end{bmatrix}\begin{bmatrix} u_L \\ d_L \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ W^{\dagger\mu} u_L \end{bmatrix} = \frac{W^{\dagger\mu}}{\sqrt{2}}\begin{bmatrix} 0 \\ u_L \end{bmatrix}$$

Notice how the u_L and d_L fields switch positions in the vector. This is what will subsequently allow interactions between these fields via the gauge bosons W^{μ}.

Lastly,

$$\frac{1}{2}\tau_3 W_3^\mu \begin{bmatrix} u_L \\ d_L \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} W_3^\mu u_L \\ W_3^\mu d_L \end{bmatrix} = \frac{1}{2}\begin{bmatrix} W_3^\mu u_L \\ -W_3^\mu d_L \end{bmatrix} = \frac{W_3^\mu}{2}\begin{bmatrix} u_L \\ -d_L \end{bmatrix}$$

Now we're ready to get back to the term from the Lagrangian.

$$\frac{g}{2}\begin{bmatrix} \bar{u}_L & \bar{d}_L \end{bmatrix}\gamma_\rho \vec{\tau}\cdot\vec{W}^\rho \begin{bmatrix} u_L \\ d_L \end{bmatrix}$$

$$= g\begin{bmatrix} \bar{u}_L & \bar{d}_L \end{bmatrix}\gamma_\rho\left(\frac{1}{2}\tau_+(W_1^\rho - iW_2^\rho) + \frac{1}{2}\tau_-(W_1^\rho + iW_2^\rho) + \frac{1}{2}\tau_3 W_3^\rho\right)\begin{bmatrix} u_L \\ d_L \end{bmatrix}$$

$$= g\begin{bmatrix} \bar{u}_L & \bar{d}_L \end{bmatrix}\gamma_\rho\left(\frac{1}{\sqrt{2}}\tau_+ W^\rho + \frac{1}{\sqrt{2}}\tau_- W^{\dagger\rho} + \frac{1}{2}\tau_3 W_3^\rho\right)\begin{bmatrix} u_L \\ d_L \end{bmatrix}$$

$$= g\begin{bmatrix} \bar{u}_L & \bar{d}_L \end{bmatrix}\gamma_\rho\left(\frac{1}{\sqrt{2}}W^\rho\begin{bmatrix} d_L \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}W^{\dagger\rho}\begin{bmatrix} 0 \\ u_L \end{bmatrix} + \frac{1}{2}W_3^\rho\begin{bmatrix} u_L \\ -d_L \end{bmatrix}\right)$$

$$= \frac{1}{\sqrt{2}}g\begin{bmatrix} \bar{u}_L & \bar{d}_L \end{bmatrix}\begin{bmatrix} W^\rho d_L \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}g\begin{bmatrix} \bar{u}_L & \bar{d}_L \end{bmatrix}\begin{bmatrix} 0 \\ W^{\dagger\rho} u_L \end{bmatrix} + \frac{1}{2}g\begin{bmatrix} \bar{u}_L & \bar{d}_L \end{bmatrix}\begin{bmatrix} W_3^\rho u_L \\ -W_3^\rho d_L \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}g\,\bar{u}_L W^\rho d_L + \frac{1}{\sqrt{2}}g\,\bar{d}_L W^{\dagger\rho} u_L + \frac{1}{2}g\,\bar{u}_L W_3^\rho u_L + \frac{1}{2}g\,\bar{d}_L(-W_3^\rho d_L)$$

$$= \frac{1}{\sqrt{2}}g\,u_L^\dagger\gamma_0\gamma_\rho W^\rho d_L + \frac{1}{\sqrt{2}}g\,d_L^\dagger\gamma_0\gamma_\rho W^{\dagger\rho} u_L + \frac{1}{2}g\,u_L^\dagger\gamma_0\gamma_\rho W_3^\rho u_L - \frac{1}{2}g\,d_L^\dagger\gamma_0\gamma_\rho W_3^\rho d_L$$

$$= \frac{1}{\sqrt{2}}g\,u^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho W^\rho\left(\frac{1-\gamma_5}{2}\right)d + \frac{1}{\sqrt{2}}g\,d^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho W^{\dagger\rho}\left(\frac{1-\gamma_5}{2}\right)u$$

$$+ \frac{1}{2}g\,u^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho W_3^\rho\left(\frac{1-\gamma_5}{2}\right)u - \frac{1}{2}g\,d^\dagger\left(\frac{1-\gamma_5}{2}\right)\gamma_0\gamma_\rho W_3^\rho\left(\frac{1-\gamma_5}{2}\right)d$$

$$= \frac{1}{\sqrt{2}}g\,\bar{u}\gamma_\rho W^\rho\left(\frac{1-\gamma_5}{2}\right)d + \frac{1}{\sqrt{2}}g\,\bar{d}\gamma_\rho W^{\dagger\rho}\left(\frac{1-\gamma_5}{2}\right)u$$

$$+ \frac{1}{2}g\,\bar{u}\gamma_\rho W_3^\rho\left(\frac{1-\gamma_5}{2}\right)u - \frac{1}{2}g\,\bar{d}\gamma_\rho W_3^\rho\left(\frac{1-\gamma_5}{2}\right)d$$

It is important to note that, while these terms do describe quark interactions, the vertex factors here are not in their final form. There are still the CKM matrix elements that govern the strength of the interactions to deal with. This requires consideration of the Higgs field and is accomplished in my "Quark Mixing" section.

II.C. The B^ρ Field

$$[\bar{u}_L \ \bar{d}_L] \frac{g'y}{2} \gamma_\rho B^\rho \begin{bmatrix} u_L \\ d_L \end{bmatrix}$$

$$= \frac{g'y}{2} [\bar{u}_L \ \bar{d}_L] \begin{bmatrix} \not{B} u_L \\ \not{B} d_L \end{bmatrix}$$

$$= \frac{g'y}{2} [\bar{u}_L \not{B} u_L + \bar{d}_L \not{B} d_L]$$

$$= \frac{g'y}{2} [u_L^\dagger \gamma_0 \gamma_\rho B^\rho u_L + d_L^\dagger \gamma_0 \gamma_\rho B^\rho d_L]$$

$$= \frac{g'y}{2} \left[u^\dagger \left(\frac{1-\gamma_5}{2}\right) \gamma_0 \gamma_\rho B^\rho \left(\frac{1-\gamma_5}{2}\right) u + d^\dagger \left(\frac{1-\gamma_5}{2}\right) \gamma_0 \gamma_\rho B^\rho \left(\frac{1-\gamma_5}{2}\right) d\right]$$

$$= \frac{g'y}{2} \left[\bar{u} \gamma_\rho \left(\frac{1-\gamma_5}{2}\right) B^\rho u + \bar{d} \gamma_\rho \left(\frac{1-\gamma_5}{2}\right) B^\rho d\right]$$

III. Find the Z^0 and Photon Interactions

The next task is to mix these terms with the W_3^μ terms from before to yield the photon and Z^0-boson interactions with ~~leptons and their neutrinos.~~ the quarks.

Recall I found in "SU(2)×U(1) Local Spontaneously Broken Symmetry" that the mixing is given by

$$\begin{bmatrix} Z^\mu \\ A^\mu \end{bmatrix} = \begin{bmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{bmatrix} \begin{bmatrix} W_3^\mu \\ B^\mu \end{bmatrix}$$

$$\text{where} \begin{cases} \cos\theta_W = \dfrac{g}{\sqrt{g^2+g'^2}} \\[2ex] \sin\theta_W = \dfrac{g'}{\sqrt{g^2+g'^2}} \end{cases}$$

Or inverted as

$$\begin{bmatrix} W_3^\mu \\ B^\mu \end{bmatrix} = \begin{bmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{bmatrix} \begin{bmatrix} Z^\mu \\ A^\mu \end{bmatrix}$$

Gather the B^ρ terms from (II.C), the right-handed B^ρ terms from (I), and the two W_3^μ terms from (II.B). Be sure to get the signs correctly from (II.B) and (II.C)

$$-\frac{g'y}{2}\bar{u}\gamma_\rho B^\rho\left(\frac{1+\gamma_5}{2}\right)u - \frac{g'y}{2}\bar{d}\gamma_\rho B^\rho\left(\frac{1+\gamma_5}{2}\right)d \qquad \text{(I)}$$

$$-\frac{1}{2}g\bar{u}\gamma_\rho W_3^\rho\left(\frac{1-\gamma_5}{2}\right)u + \frac{1}{2}g\bar{d}\gamma_\rho W_3^\rho\left(\frac{1-\gamma_5}{2}\right)d \qquad \text{(II.B)}$$

$$-\frac{g'y}{2}\left[\bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)B^\rho u + \bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)B^\rho d\right] \qquad \text{(II.C)}$$

$$\cancel{= -\frac{g'y}{2}\bar{u}\gamma_\rho u\left(-Z^P\sin\theta_W + A^P\cos\theta_W\right)}$$

$$\cancel{-\frac{g'y}{2}\bar{d}\gamma_\rho}$$

$$= -\frac{g'y}{2}\bar{u}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)u\left(-Z^P\sin\theta_W + A^P\cos\theta_W\right)$$

$$-\frac{g'y}{2}\bar{d}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)d\left(-Z^P\sin\theta_W + A^P\cos\theta_W\right)$$

$$-\frac{g}{2}\bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)u\left(Z^P\cos\theta_W + A^P\sin\theta_W\right)$$

$$+\frac{g}{2}\bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)d\left(Z^P\cos\theta_W + A^P\sin\theta_W\right)$$

$$-\frac{g'y}{2}\bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)u\left(-Z^P\sin\theta_W + A^P\cos\theta_W\right)$$

$$-\frac{g'y}{2}\bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)d\left(-Z^P\sin\theta_W + A^P\cos\theta_W\right)$$

Use $y=\frac{1}{3}$ for u_L; $y=\frac{4}{3}$ for u_R; $y=\frac{1}{3}$ for d_L; and $y=-\frac{2}{3}$ for d_R. (AH p.340, table 22.1).

$$= -\frac{2g'}{3}\bar{u}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)u\left(-Z^P\sin\theta_W + A^P\cos\theta_W\right)$$

$$+\frac{g'}{3}\bar{d}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)d\left(-Z^P\sin\theta_W + A^P\cos\theta_W\right)$$

$$-\frac{g}{2}\bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)u\left(Z^P\cos\theta_W + A^P\sin\theta_W\right)$$

$$+\frac{g}{2}\bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)d\left(Z^P\cos\theta_W + A^P\sin\theta_W\right)$$

$$-\frac{g'}{6}\bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)u\left(-Z^P\sin\theta_W + A^P\cos\theta_W\right)$$

$$-\frac{g'}{6}\bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)d\left(-Z^P\sin\theta_W + A^P\cos\theta_W\right)$$

$$= + \frac{2g'}{3} \bar{u} \gamma_\rho \left(\frac{1+\gamma_5}{2}\right) u \left(Z^\rho \sin\theta_W\right) - \frac{2g'}{3} \bar{u} \gamma_\rho \left(\frac{1+\gamma_5}{2}\right) u \left(A^\rho \cos\theta_W\right)$$

$$- \frac{g'}{3} \bar{d} \gamma_\rho \left(\frac{1+\gamma_5}{2}\right) d \left(Z^\rho \sin\theta_W\right) + \frac{g'}{3} \bar{d} \gamma_\rho \left(\frac{1+\gamma_5}{2}\right) d \left(A^\rho \cos\theta_W\right)$$

$$+ \bar{u} \gamma_\rho \left(\frac{1-\gamma_5}{2}\right) u \, Z^\rho \left(-\frac{1}{2} g \cos\theta_W + \frac{1}{6} g' \sin\theta_W\right)$$

$$+ \bar{u} \gamma_\rho \left(\frac{1-\gamma_5}{2}\right) u \, A^\rho \left(-\frac{1}{2} g \sin\theta_W - \frac{1}{6} g' \cos\theta_W\right)$$

$$+ \bar{d} \gamma_\rho \left(\frac{1-\gamma_5}{2}\right) d \, Z^\rho \left(\frac{1}{2} g \cos\theta_W + \frac{g'}{6} \sin\theta_W\right)$$

$$+ \bar{d} \gamma_\rho \left(\frac{1-\gamma_5}{2}\right) d \, A^\rho \left(\frac{1}{2} g \sin\theta_W - \frac{1}{6} g' \cos\theta_W\right)$$

Use $g' = g \dfrac{\sin\theta_W}{\cos\theta_W}$ in all terms.

$$= \frac{2}{3} \bar{u} \gamma_\rho \left(\frac{1+\gamma_5}{2}\right) u \, Z^\rho \left(g \frac{\sin\theta_W}{\cos\theta_W}\right) \sin\theta_W - \frac{2}{3} \bar{u} \gamma_\rho \left(\frac{1+\gamma_5}{2}\right) u \, A^\rho \left(g \frac{\sin\theta_W}{\cancel{\cos\theta_W}}\right) \cancel{\cos\theta_W}$$

$$- \frac{1}{3} \bar{d} \gamma_\rho \left(\frac{1+\gamma_5}{2}\right) d \, Z^\rho \left(g \frac{\sin\theta_W}{\cancel{\cos\theta_W}} \sin\theta_W\right) + \frac{1}{3} \bar{d} \gamma_\rho \left(\frac{1+\gamma_5}{2}\right) d \, A^\rho \left(g \frac{\sin\theta_W}{\cancel{\cos\theta_W}} \cancel{\cos\theta_W}\right)$$

$$+ \bar{u} \gamma_\rho \left(\frac{1-\gamma_5}{2}\right) u \, Z^\rho \left(-\frac{1}{2} g \cos\theta_W + \frac{1}{6} g \frac{\sin\theta_W}{\cos\theta_W} \sin\theta_W\right)$$

$$+ \bar{u} \gamma_\rho \left(\frac{1-\gamma_5}{2}\right) u \, A^\rho \left(-\frac{1}{2} g \sin\theta_W - \frac{1}{6} g \frac{\sin\theta_W}{\cancel{\cos\theta_W}} \cancel{\cos\theta_W}\right)$$

$$+ \bar{d} \gamma_\rho \left(\frac{1-\gamma_5}{2}\right) d \, Z^\rho \left(\frac{1}{2} g \cos\theta_W + \frac{1}{6} g \frac{\sin\theta_W}{\cos\theta_W} \sin\theta_W\right)$$

$$+ \bar{d} \gamma_\rho \left(\frac{1-\gamma_5}{2}\right) d \, A^\rho \left(\frac{1}{2} g \sin\theta_W - \frac{1}{6} g \frac{\sin\theta_W}{\cancel{\cos\theta_W}} \cancel{\cos\theta_W}\right)$$

$$= \frac{2g}{3}\bar{u}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)u\, Z^\rho\left(\frac{\sin^2\theta}{\cos\theta}\right) - \frac{2g}{3}\bar{u}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)u\, A^\rho \sin\theta_W$$

$$- \frac{g}{3}\bar{d}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)d\, Z^\rho\left(\frac{\sin^2\theta_W}{\cos\theta_W}\right) + \frac{g}{3}\bar{d}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)d\, A^\rho \sin\theta_W$$

$$+ \bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)u\, Z^\rho\left(-\frac{g}{2}\frac{\cos^2\theta_W}{\cos\theta_W} + \frac{g}{6}\frac{\sin^2\theta_W}{\cos\theta_W}\right)$$

$$+ \bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)u\, A^\rho\left(-\frac{2}{3}g\sin\theta_W\right)$$

$$+ \bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)d\, Z^\rho\left(\frac{g}{2}\frac{\cos^2\theta_W}{\cos\theta_W} + \frac{g}{6}\frac{\sin^2\theta_W}{\cos\theta_W}\right)$$

$$+ \bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)d\, A^\rho\left(\frac{1}{3}g\sin\theta_W\right)$$

$$= \frac{2g}{3}\bar{u}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)u\, Z^\rho\left(\frac{\sin^2\theta_W}{\cos\theta_W}\right) - \frac{2g}{3}\bar{u}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)u\, A^\rho \sin\theta_W$$

$$- \frac{g}{3}\bar{d}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)d\, Z^\rho\left(\frac{\sin^2\theta_W}{\cos\theta_W}\right) + \frac{g}{3}\bar{d}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)d\, A^\rho \sin\theta_W$$

$$+ \frac{g}{2\cos\theta_W}\bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)u\, Z^\rho\left(-\cos^2\theta_W + \frac{1}{3}\sin^2\theta_W\right)$$

$$- \frac{2g}{3}\bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)u\, A^\rho\left(\sin\theta_W\right)$$

$$+ \frac{g}{2\cos\theta_W}\bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)d\, Z^\rho\left(\cos^2\theta_W + \frac{1}{3}\sin^2\theta_W\right)$$

$$+ \frac{g}{3}\bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)d\, A^\rho\left(\sin\theta_W\right)$$

Using the following relations:

$$-\cos^2\theta_W + \frac{1}{3}\sin^2\theta_W = -1 + \frac{4}{3}\sin^2\theta_W$$

$$\cos^2\theta_W + \frac{1}{3}\sin^2\theta_W = 1 - \frac{2}{3}\sin^2\theta_W$$

$$\left(\frac{1-\gamma_5}{2}\right) + \left(\frac{1+\gamma_5}{2}\right) = 1$$

we get

$$= \frac{g}{2\cos\theta_W}\bar{u}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)u\, Z^\rho\left(\frac{4}{3}\sin^2\theta_W\right)$$

$$- \frac{g}{2\cos\theta_W}\bar{d}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)d\, Z^\rho\left(\frac{2}{3}\sin^2\theta_W\right)$$

$$- \frac{2g}{3}\bar{u}\gamma_\rho u A^\rho \sin\theta_W + \frac{g}{3}\bar{d}\gamma_\rho d A^\rho \sin\theta_W$$

$$+ \frac{g}{2\cos\theta_W}\bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)u\, Z^\rho\left(-1+\frac{4}{3}\sin^2\theta_W\right)$$

$$+ \frac{g}{2\cos\theta_W}\bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)d\, Z^\rho\left(1-\frac{2}{3}\sin^2\theta_W\right)$$

As usual, the electric charge is

$$e_0 = g\sin\theta_W$$

The entire Lagrangian density is

$$\mathcal{L} = i\bar{u}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)\partial^\rho u + i\bar{d}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)\partial^\rho d + i\bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho u + i\bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)\partial^\rho d$$

$$+ \frac{1}{\sqrt{2}}g\bar{u}\gamma_\rho W^\rho\left(\frac{1-\gamma_5}{2}\right)d + \frac{1}{\sqrt{2}}g\bar{d}\gamma_\rho W^{\dagger\rho}\left(\frac{1-\gamma_5}{2}\right)u$$

$$+ \frac{g\, Z^\rho}{2\cos\theta_W}\Big[\bar{u}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)u\left(\frac{4}{3}\sin^2\theta_W\right) - \bar{d}\gamma_\rho\left(\frac{1+\gamma_5}{2}\right)d\left(\frac{2}{3}\sin^2\theta_W\right)$$

$$+ \bar{u}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)u\left(-1+\frac{4}{3}\sin^2\theta_W\right) - \bar{d}\gamma_\rho\left(\frac{1-\gamma_5}{2}\right)d\left(-1+\frac{2}{3}\sin^2\theta_W\right)\Big]$$

$$- \frac{2e_0}{3}\bar{u}\gamma_\rho u A^\rho + \frac{e_0}{3}\bar{d}\gamma_\rho d A^\rho$$