The Higgs Sector of the Standard Model

We saw in "SU(2) x U(1) Local Spontaneously Broken Symmetry" that the Lagrangian for the Higgs Sector is

$$\mathcal{L}_{\text{Higgs}} = (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \frac{\lambda}{4}(\phi^\dagger \phi)^2 - \frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$

where
$$\begin{cases} D_\mu = \partial_\mu + \frac{ig}{2}\vec{\tau}\cdot\vec{W}_\mu + \frac{ig'y}{2}B_\mu \\ \vec{F}^{\mu\nu} = \partial^\mu \vec{W}^\nu - \partial^\nu \vec{W}^\mu - g\vec{W}^\mu \times \vec{W}^\nu \\ G^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu \end{cases}$$

Consider only the 1st term for now.

$$(D_\mu\phi)^\dagger (D^\mu \phi)$$
$$= \left(\partial_\mu \phi + \frac{ig}{2}\vec{\tau}\cdot\vec{W}_\mu \phi + \frac{ig'y}{2}B_\mu \phi\right)^\dagger \left(\partial^\mu \phi + \frac{ig}{2}\vec{\tau}\cdot\vec{W}^\mu \phi + \frac{ig'y}{2}B^\mu \phi\right)$$
$$= (\partial_\mu \phi)^\dagger (\partial^\mu \phi) + (\partial_\mu \phi)^\dagger \left(\frac{ig}{2}\vec{\tau}\cdot\vec{W}^\mu \phi + \frac{ig'y}{2}B^\mu \phi\right) + \left(\frac{ig}{2}\vec{\tau}\cdot\vec{W}_\mu \phi + \frac{ig'y}{2}B_\mu \phi\right)^\dagger (\partial^\mu \phi)$$
$$+ \left[\frac{ig}{2}\vec{\tau}\cdot\vec{W}_\mu \phi + \frac{ig'y}{2}B_\mu \phi\right]^\dagger \left[\frac{ig}{2}\vec{\tau}\cdot\vec{W}^\mu \phi + \frac{ig'y}{2}B^\mu \phi\right]$$

Now let's work on the 2nd line of this. Note that in my expression of the Higgs doublet I'll be skipping straight to vacuum expectation values.

$$\frac{ig}{2}\vec{\tau}\cdot\vec{W}_\mu\phi + \frac{ig'y}{2}B_\mu\phi$$

$$= \frac{ig}{2}\left[\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}W_{1\mu} + \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}W_{2\mu} + \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}W_{3\mu}\right]\begin{bmatrix}0\\ \frac{1}{\sqrt{2}}(v+H)\end{bmatrix} + \frac{ig'y}{2}B_\mu\begin{bmatrix}0\\ \frac{1}{\sqrt{2}}(v+H)\end{bmatrix}$$

$$= \frac{ig}{2}\left[\begin{bmatrix}\frac{1}{\sqrt{2}}W_{1\mu}(v+H)\\ 0\end{bmatrix} + \begin{bmatrix}\frac{-i}{\sqrt{2}}W_{2\mu}(v+H)\\ 0\end{bmatrix} + \begin{bmatrix}0\\ -\frac{1}{\sqrt{2}}W_{3\mu}(v+H)\end{bmatrix}\right] + \begin{bmatrix}0\\ \frac{ig'y}{2\sqrt{2}}B_\mu(v+H)\end{bmatrix}$$

$$= \begin{bmatrix}\frac{ig}{2\sqrt{2}}W_{1\mu}(v+H) + \frac{g}{2\sqrt{2}}W_{2\mu}(v+H)\\ -\frac{ig}{2\sqrt{2}}W_{3\mu}(v+H) + \frac{ig'y}{2\sqrt{2}}B_\mu(v+H)\end{bmatrix}$$

$$= \begin{bmatrix}\frac{ig}{2}(W_{1\mu} - iW_{2\mu})\frac{1}{\sqrt{2}}(v+H)\\ -\frac{ig}{2}W_{3\mu}\frac{1}{\sqrt{2}}(v+H) + \frac{ig'y}{2}B_\mu\frac{1}{\sqrt{2}}(v+H)\end{bmatrix}$$

Next, multiply this by its Hermitian conjugate as follows:

$$\begin{bmatrix}-\frac{ig}{2}(W_{1\mu}+iW_{2\mu})\frac{1}{\sqrt{2}}(v+H) & \frac{ig}{2}W_{3\mu}\frac{1}{\sqrt{2}}(v+H) - \frac{ig'y}{2}B_\mu\frac{1}{\sqrt{2}}(v+H)\end{bmatrix}\begin{bmatrix}\frac{ig}{2}(W_1^\nu - iW_2^\nu)\frac{1}{\sqrt{2}}(v+H)\\ -\frac{ig}{2}W_3^\nu\frac{1}{\sqrt{2}}(v+H) + \frac{ig'y}{2}B^\nu\frac{1}{\sqrt{2}}(v+H)\end{bmatrix}$$

$$= \frac{g^2}{4}|W_1 - iW_2|^2\frac{1}{2}(v+H)^2 + \frac{g^2}{4}|W_3|^2\frac{1}{2}(v+H)^2 - \frac{gg'y}{4}W_{3\mu}B^\nu\frac{1}{2}(v+H)^2$$

$$- \frac{gg'y}{4}W_3^\nu B_\mu\frac{1}{2}(v+H)^2 + \frac{g'^2y^2}{4}|B|^2\frac{1}{2}(v+H)^2$$

$$= \frac{g^2}{4} W^\dagger_\mu W^\mu v^2 + \frac{g^2}{2} W^\dagger_\mu W^\mu v H + \frac{g^2}{4} W^\dagger_\mu W^\mu H + \frac{g^2}{8} |W_3|^2 v^2 + \frac{g^2}{4} |W_3|^2 v H + \frac{g^2}{8} |W_3|^2 H^2$$

$$- \frac{g g'}{4} W_{3\mu} B^\mu v^2 - \frac{g g'}{2} W_{3\mu} B^\mu v H - \frac{g' g}{2} W_{3\mu} B^\mu H^2$$

$$+ \frac{g'^2}{8} |B|^2 v^2 + \frac{g'^2}{4} |B|^2 v H + \frac{g'^2}{8} |B|^2 H^2$$

With that done, let's piece together the terms from the 1st line.

$$(\partial_\mu \phi)^\dagger (\partial^\mu \phi) = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} (\partial_\mu v + \partial_\mu H) \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} (\partial^\mu v + \partial^\mu H) \end{bmatrix} = \frac{1}{2} (\partial_\mu H)(\partial^\mu H)$$

$$(\partial_\mu \phi)^\dagger \left(\frac{ig}{2} \vec{\tau} \cdot \vec{W}^\mu \phi + \frac{ig'}{2} B^\mu \phi \right)$$

$$= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \partial_\mu H \end{bmatrix} \left[\begin{bmatrix} \frac{ig}{2\sqrt{2}} W_1^\mu (v+H) \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{g}{2\sqrt{2}} W_2^\mu (v+H) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-ig}{2\sqrt{2}} W_3^\mu (v+H) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{ig'}{2\sqrt{2}} B^\mu (v+H) \end{bmatrix} \right]$$

$$= \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} \partial_\mu H \end{bmatrix} \begin{bmatrix} \frac{ig}{2} (W_1^\mu - i W_2^\mu) \frac{1}{\sqrt{2}} (v+H) \\ -\frac{ig}{2} W_3^\mu \frac{1}{\sqrt{2}} (v+H) + \frac{ig'}{2\sqrt{2}} B^\mu (v+H) \end{bmatrix}$$

$$= -\frac{ig}{4} (\partial_\mu H) W_3^\mu (v+H) + \frac{ig'}{4} (\partial_\mu H) B^\mu (v+H)$$

$$\left[\frac{ig}{2}\vec{\tau}\cdot\vec{W}_\mu\phi + \frac{ig'y}{2}B_\mu\phi\right]^\dagger(\partial^\mu\phi)$$

$$= \begin{bmatrix} \frac{ig}{2\sqrt{2}}(W_{1\mu} - iW_{2\mu})(v+H) \\ -\frac{ig}{2\sqrt{2}}W_{3\mu}(v+H) + \frac{ig'y}{2\sqrt{2}}B_\mu(v+H) \end{bmatrix}^\dagger \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}}\partial^\mu H \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-ig}{2\sqrt{2}}(W_{1\mu} + iW_{2\mu})(v+H) & \frac{ig}{2\sqrt{2}}W_{3\mu}(v+H) - \frac{ig'y}{2\sqrt{2}}B_\mu(v+H) \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}}\partial^\mu H \end{bmatrix}$$

$$= \frac{ig}{4}W_{3\mu}(v+H)\partial^\mu H - \frac{ig'y}{4}B_\mu(v+H)\partial^\mu H$$

Finally, let's put the 1st term of the Lagrangian back together.

$$(D_\mu\phi)^\dagger(D^\mu\phi)$$

$$= \frac{1}{2}(\partial_\mu H)(\partial^\mu H) - \frac{ig}{4}(\partial_\mu H)W_3^\mu(v+H) + \frac{ig'y}{4}(\partial_\mu H)B^\mu(v+H)$$

$$+ \frac{ig}{4}W_{3\mu}(v+H)(\partial^\mu H) - \frac{ig'y}{4}B_\mu(v+H)(\partial^\mu H)$$

$$+ \frac{g^2}{4}W_\mu^\dagger W^\mu v^2 + \frac{g^2}{2}W_\mu^\dagger W^\mu vH + \frac{g^2}{4}W_\mu^\dagger W^\mu H^2$$

$$+ \frac{g^2}{8}|W_3|^2v^2 + \frac{g^2}{4}|W_3|^2vH + \frac{g^2}{8}|W_3|^2H^2$$

$$- \frac{gg'y}{4}W_{3\mu}B^\mu v^2 - \frac{gg'y}{2}W_{3\mu}B^\mu vH - \frac{gg'y}{2}W_{3\mu}B^\mu H^2$$

$$+ \frac{g'^2y}{8}|B|^2v^2 + \frac{g'^2y^2}{4}|B|^2vH + \frac{g'^2y}{8}|B|^2H^2$$

Let's move on to the other terms of the Lagrangian.

$$\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} = (\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu - g\vec{W}_\mu\times\vec{W}_\nu)\cdot(\partial^\mu\vec{W}^\nu - \partial^\nu\vec{W}^\mu - g\vec{W}^\mu\times\vec{W}^\nu)$$

$$= (\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu)\cdot(\partial^\mu\vec{W}^\nu - \partial^\nu\vec{W}^\mu) - g(\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu)\cdot(\vec{W}^\mu\times\vec{W}^\nu)$$

$$- g(\vec{W}_\mu\times\vec{W}_\nu)\cdot(\partial^\mu\vec{W}^\nu - \partial^\nu\vec{W}^\mu) + g^2(\vec{W}_\mu\times\vec{W}_\nu)\cdot(\vec{W}^\mu\times\vec{W}^\nu)$$

$$= (\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu)\cdot(\partial^\mu\vec{W}^\nu - \partial^\nu\vec{W}^\mu) - 2g(\vec{W}_\mu\times\vec{W}_\nu)\cdot(\partial^\mu\vec{W}^\nu - \partial^\nu\vec{W}^\mu)$$

$$+ g^2\left[|W_\mu|^2|W_\nu|^2 - |\vec{W}_\mu\cdot\vec{W}_\nu|^2\right]$$

Now we're ready to put the Lagrangian back together.

$$\mathcal{L}_{\text{Higgs}} = (D_\mu\phi)^\dagger(D^\mu\phi) + \mu^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 - \frac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$

$$\langle 0|\mathcal{L}_{\text{Higgs}}|0\rangle = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{g^2v^2}{4}W_\mu^\dagger W^\mu + \frac{g^2v}{2}W_\mu^\dagger W^\mu H + \frac{g^2}{4}W_\mu^\dagger W^\mu H^2$$

~~$\frac{g^2v^2}{8}|W_1 - iW_2|^2 + \frac{g^2v}{4}|W_1 - iW_2|^2 H + \frac{g^2}{8}|W_1 - iW_2|^2 H^2$~~

$$+ \frac{g^2v^2}{8}|W_3|^2 + \frac{g^2v}{4}|W_3|^2 H + \frac{g^2}{8}|W_3|^2 H^2$$

$$- \frac{gg'yv^2}{4}W_{3\mu}B^\mu - \frac{gg'yv}{2}W_{3\mu}B^\mu H - \frac{gg'y}{2}W_{3\mu}B^\mu H^2$$

$$+ \frac{g'^2y^2v^2}{8}|B|^2 + \frac{g'^2y^2v}{4}|B|^2 H + \frac{g'^2y^2}{8}|B|^2 H^2$$

$$+ \frac{1}{2}\mu^2(v+H)^2 - \frac{\lambda}{16}(v+H)^4$$

$$- \frac{1}{4}(\partial_\mu\vec{W}_\nu - \partial_\nu\vec{W}_\mu)\cdot(\partial^\mu\vec{W}^\nu - \partial^\nu\vec{W}^\mu) + \frac{1}{2}g(\vec{W}_\mu\times\vec{W}_\nu)\cdot(\partial^\mu\vec{W}^\nu - \partial^\nu\vec{W}^\mu)$$

$$- \frac{1}{4}g^2\left[|\vec{W}_\mu|^2|\vec{W}_\nu|^2 - |\vec{W}_\mu\cdot\vec{W}_\nu|^2\right] - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$

$$= \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{1}{2}\mu^2(v+H)^2 - \frac{\lambda}{16}(v+H)^4 + \frac{g^2v}{4} W_\mu^\dagger W^\mu + \frac{g^2 v}{2} W_\mu^\dagger W^\mu H + \frac{g^2}{4} W_\mu^\dagger W^\mu H^2$$

~~$+ \frac{g^2v^2}{8}|W_1|^2 + \frac{g^2v^2}{8}|W_2|^2 + \frac{g^2v}{4}|W_1|^2 H + \frac{g^2v}{4}|W_2|^2 H + \frac{g^2}{8}|W_1|^2 H^2 + \frac{g^2}{8}|W_2|^2 H^2$~~

$$+ \frac{g^2v^2}{8}|W_3|^2 + \frac{g^2v}{4}|W_3|^2 H + \frac{g^2}{8}|W_3|^2 H^2$$

$$- \frac{gg'yv^2}{4} W_{3\mu} B^\mu - \frac{gg'yv}{2} W_{3\mu} B^\mu H - \frac{gg'y}{2} W_{3\mu} B^\mu H^2$$

$$+ \frac{g'^2 y^2 v^2}{8}|B|^2 + \frac{g'^2 y^2 v}{4}|B|^2 H + \frac{g'^2 y^2}{8}|B|^2 H^2$$

$$- \frac{1}{4}(\partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu)\cdot(\partial^\mu \vec{W}^\nu - \partial^\nu \vec{W}^\mu) + g(\vec{W}_\mu \times \vec{W}_\nu)\cdot(\partial^\mu \vec{W}^\nu - \partial^\nu \vec{W}^\mu)$$

$$- \frac{1}{4}g^2\left[|\vec{W}_\mu|^2|\vec{W}_\nu|^2 - |\vec{W}_\mu \cdot \vec{W}_\nu|^2\right] - \frac{1}{4} G_{\mu\nu} G^{\mu\nu}$$

Consider the boxed terms:

$$\frac{g^2v^2}{8}|W_3|^2 - \frac{gg'yv^2}{4} W_{3\mu} B^\mu + \frac{g'^2 y^2 v^2}{8}|B|^2$$

$$= \frac{v^2}{8}\left[g^2|W_3|^2 - 2gg'y W_{3\mu} B^\mu + g'^2 y^2 |B|^2\right]$$

$$= \frac{v^2}{8}\left[(gW_{3\mu} - g'yB_\mu)(gW_3^\mu - g'yB^\mu)\right]$$

Now we may write the Lagrangian in the form of (19.88) Aitchison + Hey.

$$\langle 0|\mathcal{L}_{\text{Higgs}}|0\rangle = \frac{1}{2}(\partial_\mu H)(\partial^\mu H) + \frac{1}{2}\mu^2(v+H)^2 - \frac{\lambda}{16}(v+H)^4$$

$$- \frac{1}{4}(\partial_\mu W_{1\nu} - \partial_\nu W_{1\mu})(\partial^\mu W_1^\nu - \partial^\nu W_1^\mu)$$

$$- \frac{1}{4}(\partial_\mu W_{2\nu} - \partial_\nu W_{2\mu})(\partial^\mu W_2^\nu - \partial^\nu W_2^\mu)$$

$$- \frac{1}{4}(\partial_\mu W_{3\nu} - \partial_\nu W_{3\mu})(\partial^\mu W_3^\nu - \partial^\nu W_3^\mu) - \frac{1}{4}G_{\mu\nu}G^{\mu\nu}$$

$$+ \frac{1}{8}v^2(gW_{3\mu} - g'yB_\mu)(gW_3^\mu - g'yB^\mu)$$

$$+ \frac{g^2v^2}{4}W_\mu^\dagger W^\mu + \frac{g^2v}{2}W_\mu^\dagger W^\mu H + \frac{g^2}{4}W_\mu^\dagger W^\mu H^2 + \frac{g^2v}{4}|W_3|^2 H + \frac{g^2}{8}|W_3|^2 H^2$$

$$- \frac{gg'yv}{2}W_{3\mu}B^\mu H - \frac{gg'y}{2}W_{3\mu}B^\mu H^2 + \frac{g'^2y^2v}{4}|B|^2 H + \frac{g'^2y^2}{8}|B|^2 H^2$$

$$+ \frac{1}{2}g(\vec{W}_\mu \times \vec{W}_\nu)\cdot(\partial^\mu \vec{W}^\nu - \partial^\nu \vec{W}^\mu) - \frac{1}{4}g^2\left[|\vec{W}_\mu|^2|\vec{W}_\nu|^2 - |\vec{W}_\mu\cdot\vec{W}_\nu|^2\right]$$

Let's pause to examine the terms. 1st line: Higgs kinetic energy, mass, and self-interaction. 2nd and 3rd lines: the $W^\pm$ (via $W_{1,2}$) kinetic terms. 4th and 5th lines: This shows that the W_3 and B gauge fields are mixed. After unmixing, these become the Z^0 and photon. 6th and 7th lines: $W^\pm$ mass terms, trilinear and quadralinear couplings among the gauge fields and Higgs. 8th line: Quadralinear couplings among the gauge bosons only.

Unmix the W_3 and B gauge fields with

$$\begin{cases} B^\mu = A^\mu \cos\theta_W - Z^\mu \sin\theta_W \\ W_3^\mu = Z^\mu \cos\theta_W + A^\mu \sin\theta_W \end{cases}$$

In "SU(2)×U(1) Local Spontaneously Broken Symmetry" I found that the 4th and 5th lines become

$$-\frac{1}{4}\left(Z_{\mu\nu}Z^{\mu\nu}+\mathcal{A}_{\mu\nu}\mathcal{A}^{\mu\nu}\right)+\frac{1}{8}v^2 Z_\mu Z^\mu\left(g^2+g'^2\right)$$

$$\text{where}\quad\begin{cases} Z_{\mu\nu}\equiv\partial_\mu Z_\nu-\partial_\nu Z_\mu \\ \mathcal{A}_{\mu\nu}\equiv\partial_\mu A_\nu-\partial_\nu A_\mu \end{cases}$$

So refer to that for the work.

The vertex factors for the trilinear and quadrilinear couplings can be found by making the same substitutions into the appropriate terms. I'll do just a few for now, to get trilinear couplings for unmixed Z^0 and photon.

$$\frac{g^2v}{4}|W_3|^2H-\frac{gg'yv}{2}W_{3\mu}B^\mu H+\frac{g'^2y^2v}{4}|B|^2H$$

$$=\frac{g^2v}{4}\left(Z_\mu\cos\theta_w+A_\mu\sin\theta_w\right)\left(Z^\mu\cos\theta_w+A^\mu\sin\theta_w\right)H$$

$$-\frac{gg'yv}{2}\left(Z_\mu\cos\theta_w+A_\mu\sin\theta_w\right)\left(A^\mu\cos\theta_w-Z^\mu\sin\theta_w\right)H$$

$$+\frac{g'^2y^2v}{4}\left(A_\mu\cos\theta_w-Z^\mu\sin\theta_w\right)\left(A^\mu\cos\theta_w-Z^\mu\sin\theta_w\right)H$$

$$=\frac{g^2v}{4}\left(Z_\mu Z^\mu H\cos^2\theta_w+2Z_\mu A^\mu H\sin\theta_w\cos\theta_w+A_\mu A^\mu H\sin^2\theta_w\right)$$

$$-\frac{gg'yv}{2}\left(Z_\mu A^\mu H\cos^2\theta_w-Z_\mu Z^\mu H\sin\theta_w\cos\theta_w+A_\mu A^\mu H\sin\theta_w\cos\theta_w-A_\mu Z^\mu H\sin^2\theta_w\right)$$

$$+\frac{g'^2yv}{4}\left(A_\mu A^\mu H\cos^2\theta_w-2A_\mu Z^\mu H\sin\theta_w\cos\theta_w+Z_\mu Z^\mu H\sin^2\theta_w\right)$$

$$= Z_\mu Z^\mu H\left(\frac{g^2 v}{4}\cos^2\theta + \frac{g g' v}{2}\sin\theta_w\cos\theta_w + \frac{g'^2 v}{4}\sin^2\theta_w\right)$$

$$+ Z_\mu A^\mu H\left(\frac{g^2 v}{2}\sin\theta_w\cos\theta_w - \frac{g g' v}{2}\cos^2\theta_w + \frac{g g' v}{2}\sin^2\theta_w - \frac{g'^2 v}{2}\sin\theta_w\cos\theta_w\right)$$

$$+ A_\mu A^\mu H\left(\frac{g^2 v}{4}\sin^2\theta_w - \frac{g g' v}{2}\sin\theta_w\cos\theta_w + \frac{g'^2 v}{4}\cos^2\theta_w\right)$$

$$\begin{cases} \cos\theta_w = \dfrac{g}{\sqrt{g^2 + g'^2}} \\ \sin\theta_w = \dfrac{g'}{\sqrt{g^2 + g'^2}} \end{cases}$$

Simplify vertex factors if necessary.